University of Global Village (UGV), Barishal
Dept. of Electrical and Electronic Engineering (EEE)

Signals and Systems
EEE 0714-3201

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'Imagination is more important than knowledge'
– Albert Einstein

Basic Course Information

Course Title	Signals & Systems
Course Code	EEE- 0714-3201
Credits	03
CIE Marks	90
SEE Marks	60
Exam Hours	2 hours (Mid Exam) 3 hours (Semester Final Exam)
Level	6^{th} Semester
Academic Session	Summer 2025

Course Name: Signals & Systems(EEE 0714-3201)

3 Credit Course

Class:	**17 weeks (2 classes per week) =34 Hours**
Preparation Leave (PL):	02 weeks
Exam:	04 weeks
Results:	02 weeks
Total:	**25 Weeks**

Attendance:

Students with more than or equal to 80% attendance in this course will be eligible to sit for the Semester End Examination (SEE). SEE is mandatory for all students.

Continuous Assessment Strategy

Quizzes

Altogether 4 quizzes may be taken during the semester, 2 quizzes will be taken for midterm and 2 quizzes will be taken for final term.

Assignment

Altogether 2 assignments may be taken during the semester, 1 assignments will be taken for midterm and 1 assignments will be taken for final term.

Presentation

The students will have to form a group of maximum 3 members. The topic of the presentation will be given to each group and students will have to do the group presentation on the given topic.

ASSESSMENT PATTERN

CIE- Continuous Internal Evaluation (90 Marks)

Bloom's Category Marks (out of 90)	Tests (45)	Quizzes (15)	External Participation in Curricular/Co-Curricular Activities (15)
Remember	08	08	Bloom's Affective Domain: (Attitude or will) Attendance: 15 Copy or attempt to copy: −10 Late Assignment: -10
Understand	08	07	
Apply	08		
Analyze	08		
Evaluate	08		
Create	05		

SEE- Semester End Examination (60 Marks)

Bloom's Category	Tests
Remember	**10**
Understand	**10**
Apply	**10**
Analyze	**10**
Evaluate	**10**
Create	**10**

Course Learning Outcome (CLO)

Serial No.	Course Learning Outcome (CLO)	**Blooms Taxonomy Level**
CLO-1	**Understand** the concept of signals and systems in time, frequency and Laplace domain	**1,2 Remembering, Understanding**
CLO-2	**Explain** different properties of systems and signals	**3 Applying**
CLO-3	**Analize** responses of LTI systems for different applications	**4 Analyzing**
CLO-4	**Investigate** the stability of LTI systems	**1,2,5 Remembering, Understanding, Creating**

SYNOPSIS / RATIONALE

This course lays the foundation for understanding how signals (functions that convey information) interact with systems (entities that process these signals). It explores both continuous-time and discrete-time signal representations, system properties (such as linearity and stability), and mathematical tools like convolution, Fourier series, Laplace transforms, and state-space methods. The concepts introduced are central to advanced topics such as Digital Signal Processing, Control Systems, and Communication Systems.

Course Objective

- **Analyze** and **classify** various types of signals and systems using mathematical and graphical representations.
- **Determine** and **verify** system properties such as linearity, causality, stability, and memory.
- **Solve** system differential equations using analytical techniques including zero-input and zero-state responses.
- **Apply** Fourier and Laplace transforms to **analyze** system behavior in time and frequency domains.
- **Evaluate** system response using impulse response, transfer functions, and state-space representation.

COURSE OUTLINE

Sl.	Content of Course	Hrs	CLOs
1	Classification of signals, basic operation on signals, Elementary signals, representation of signals using impulse function, Systems- classification	**10**	CLO1 CLO2
2	Linearity, causality of LTI, Time invariance, memory, Stability, invertibility, Stability - system representation	**10**	CLO2
3	Order of the system, Solution techniques, Zero state and zero input response, Impulse response- convolution integral, Determination of system properties, State variable- basic concept, state equation and time domain solution, Fourier series- properties	**20**	CLO3
4	System response, frequency response of LTI systems, Fourier transformation- properties, System transfer function, Properties of, Laplace transformation, Inverse transform, solution of system equations, System stability and frequency response and application	**20**	CLO1 CLO4

COURSE SCHEDULE

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	CLOs
1	Introduction to Signals and Systems, Signal Classification	Lecture + Visual Examples	Class Participation, Short Quiz	CLO1
2	Basic Signal Operations (Time shifting, Scaling, Inversion)	Interactive Demo + Graphical Assignments	Homework + Spot Questions	CLO1
3	Elementary Signals: Step, Ramp, Exponential, Sinusoidal	Matlab Simulation + Graph Plotting	Assignment + Oral Explanation	CLO1
4	Impulse Representation, Dirac Delta and Sampling Functions	Theory + Simulation	Written Quiz	CLO1, CLO2
5	System Classification: Static/Dynamic, Linear/Nonlinear	Group Discussion + Venn Diagrams	Class Test	CLO2

COURSE SCHEDULE

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	CLOs
6	Linearity, Time Invariance, Causality, Memory	Case Studies + Examples	Viva + Problem Solving	CLO2
7	System Stability, Invertibility and Representations	Problem-Based Learning	Written Test	CLO2
8	Order of System and System Equation Types	Lecture + MATLAB Scripts	Homework + Coding Assignment	CLO3
9	Solution Techniques: Homogeneous/Particular Solutions	Solved Examples	Quiz	CLO3
10	Zero Input & Zero State Response	Interactive Problem Solving	Class Test	CLO3

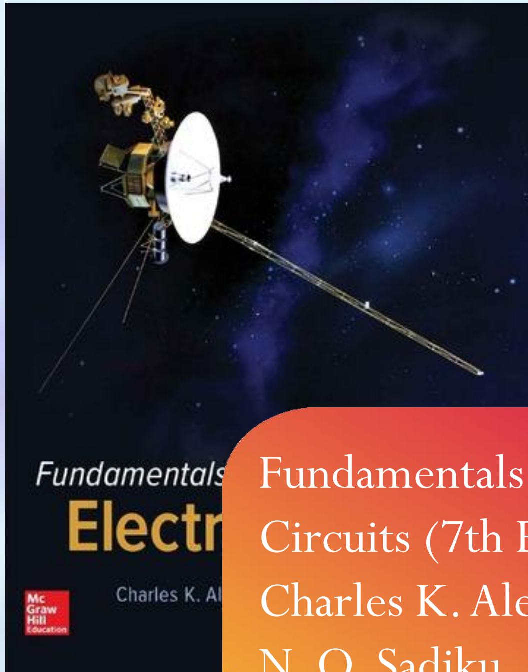
COURSE SCHEDULE

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	CLOs
11	Impulse Response and Convolution Integral	Visualization + Analytical Solutions	Problem Sheet + Viva	CLO3
12	System Properties from Impulse Response	MATLAB Analysis	Assignment + Spot Test	CLO3
13	State Variables and State Equations	Block Diagram Explanation + MATLAB	Assignment + Viva	CLO3
14	Time Domain State-Space Solution	Numerical Methods	Assignment + Quiz	CLO3
15	Fourier Series and Properties	Derivation + Simulation	Written Quiz	CLO3

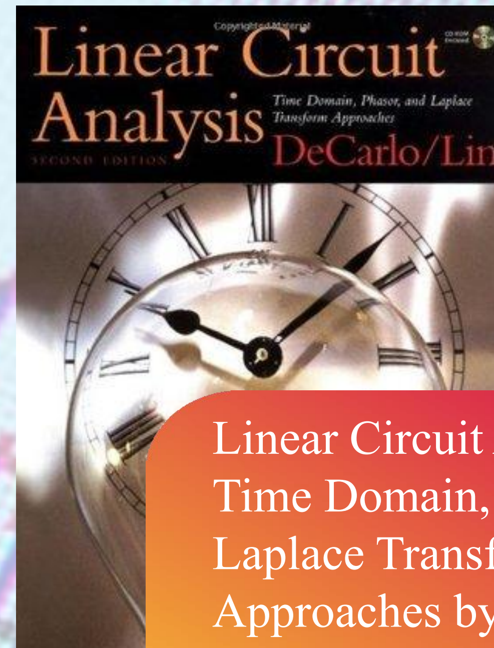
COURSE SCHEDULE

Week	Topics	Teaching-Learning Strategy	Assessment Strategy	CLOs
16	System Frequency Response, Bode Plots	Graphical and MATLAB	Class Test	CLO1, CLO4
17	Fourier Transform and Properties	Analytical Derivation + Code Demo	Mid-Term	CLO4
18	Laplace Transform: Properties and Inverse	Problem Solving + Demos	Spot Questions + Assignment	CLO4
19	Transfer Function and System Analysis	Lecture + Application Scenarios	Class Participation	CLO4
20	Final Review: Stability, Frequency Response, Applications	Concept Mapping + Open Q&A	Final Exam	CLO1, CLO4

REFERENCE BOOK



Fundamentals of Electric Circuits (7th Edition) by Charles K. Alexander & Mathew N. O. Sadiku.



Linear Circuit Analysis: Time Domain, Phasor, and Laplace Transform Approaches by Raymond A. DeCarlo

Video Lecture Playlist
https://youtube.com/playlist?list=PLgluYk4ut4L2RtIlyH42cVX6JZUUVHYzI&si=vx1s1InIlij7ywG6

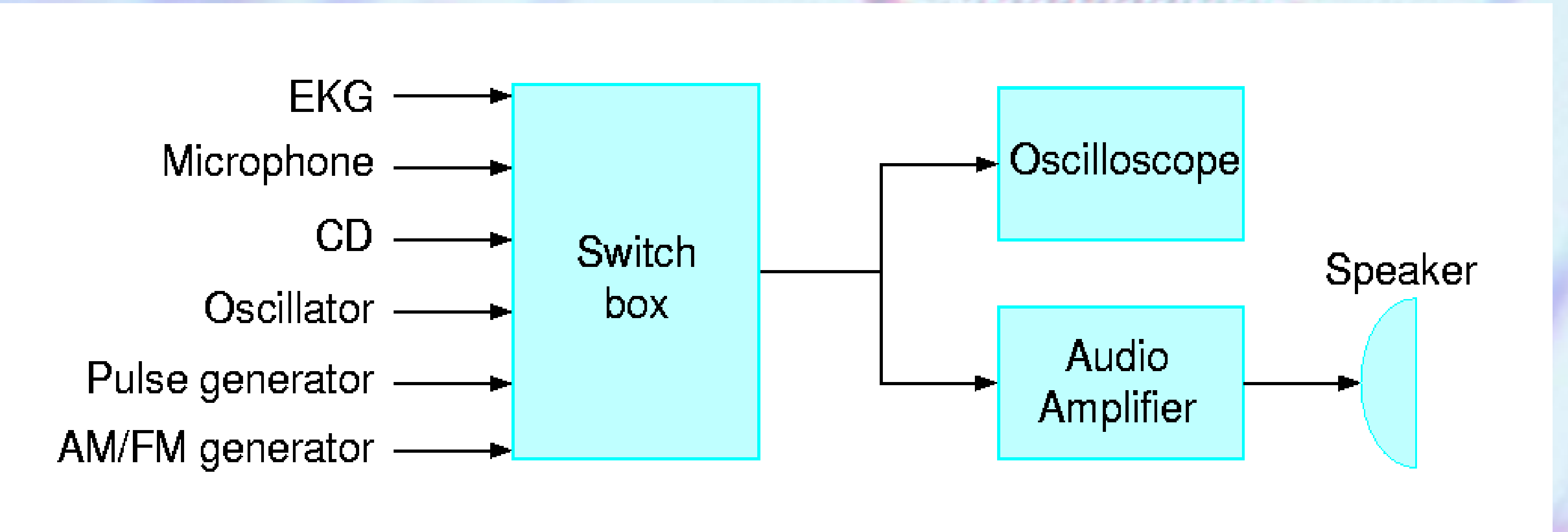
Bloom Taxonomy Cognitive Domain Action Verbs

Remembering (C1)	Choose • Define • Find • How • Label • List • Match • Name • Omit • Recall • Relate • Select • Show • Spell • Tell • What • When • Where • Which • Who • Why
Understanding (C2)	Classify • Compare • Contrast • Demonstrate • Explain • Extend • Illustrate • Infer • Interpret • Outline • Relate • Rephrase • Show • Summarize • Translate
Applying (C3)	Apply • Build • Choose • Construct • Develop • Experiment with • Identify • Interview • Make use of • Model • Organize • Plan • Select • Solve • Utilize
Analyzing (C4)	Analyze • Assume • Categorize • Classify • Compare • Conclusion • Contrast • Discover • Dissect • Distinguish • Divide • Examine • Function • Inference • Inspect • List • Motive • Relationships • Simplify • Survey • Take part in • Test for • Theme
Evaluating (C5)	Agree • Appraise • Assess • Award • Choose • Compare • Conclude • Criteria • Criticize • Decide • Deduct • Defend • Determine • Disprove • Estimate • Evaluate • Explain • Importance • Influence • Interpret • Judge • Justify • Mark • Measure • Opinion • Perceive • Prioritize • Prove • Rate • Recommend • Rule on • Select • Support • Value
Creating (C6)	Adapt • Build • Change • Choose • Combine • Compile • Compose • Construct • Create • Delete • Design • Develop • Discuss • Elaborate • Estimate • Formulate • Happen • Imagine • Improve • Invent • Make up • Maximize • Minimize • Modify • Original • Originate • Plan • Predict • Propose • Solution • Solve • Suppose • Test • Theory

Week 1

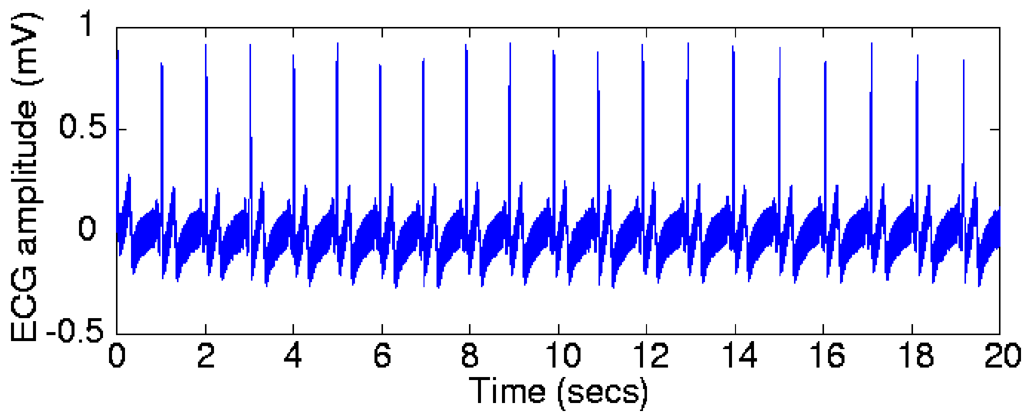
Slide 16-34

Different Types of Signals



- Type of Independent Variable

Time is often the independent variable. Example: the electrical activity of the heart recorded with chest electrodes — the electrocardiogram (ECG or EKG).

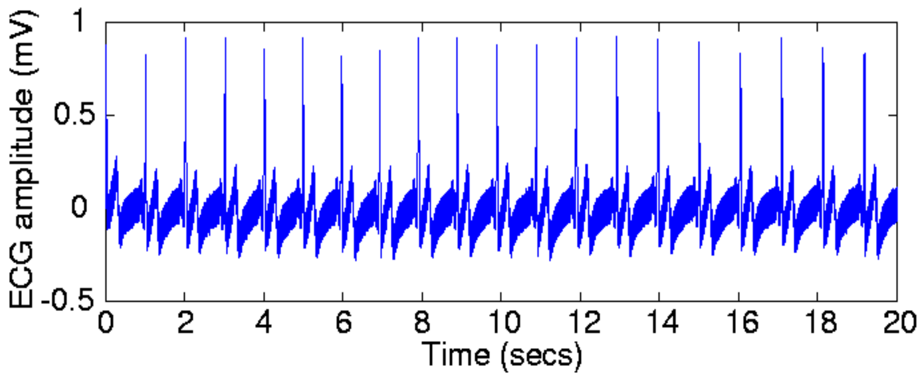


The term *time* is often used generically, to represent the independent variable of a signal. the independent variable may be a spatial variable such as in an image. Here grayscale information is specified as a function of position.

Cervical MRI

Independent Variable Dimensionality

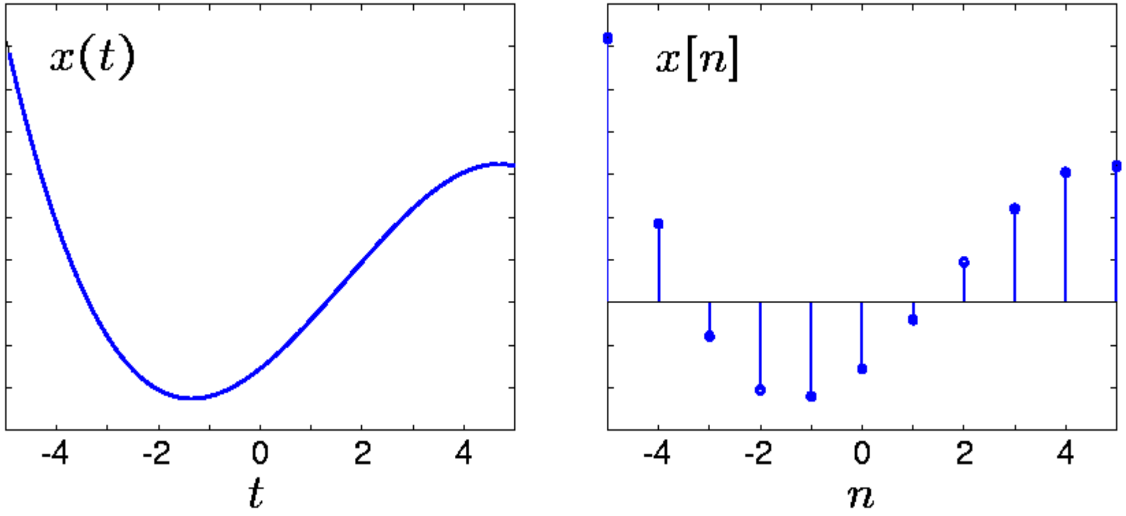
An independent variable can be 1-D (t in the EKG) or 2-D (x,y in the image).



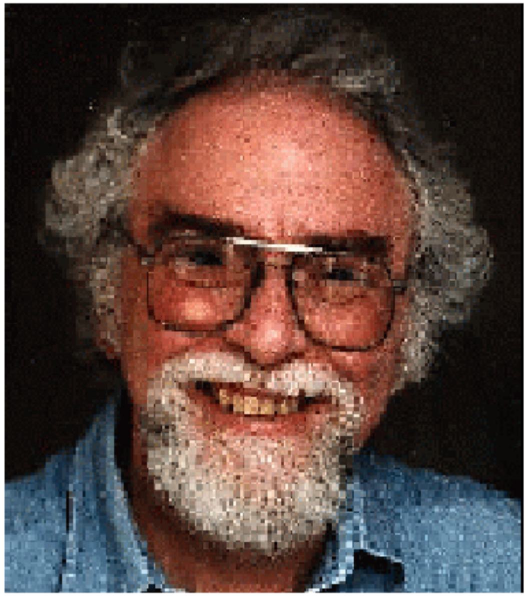
6.003 examples are mostly 1-D, but many applications use multiple dimensions (radar, MRIs, numerical simulation).

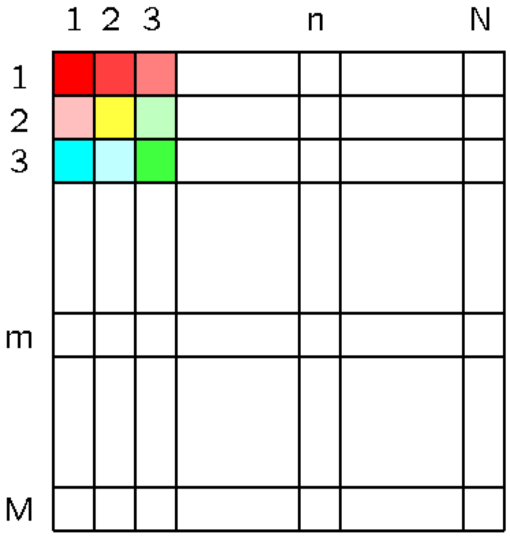
Continuous Time (CT) and Discrete-Time (DT) Signals

CT signals take on real or complex values as a function of an independent variable that ranges over the real numbers and are denoted as $x(t)$. DT signals take on real or complex values as a function of an independent variable that ranges over the integers and are denoted as $x[n]$. Note the use of parentheses for CT signals and square brackets for DT signals.



An image example on the left, its DT representation on the right





The image on the left consists of 302×435 picture elements (pixels) each of which is represented by a triplet of numbers {R,G,B} that encode the color. Thus, the signal is represented by $c[n, m]$ where m and n are the independent variables that specify pixel location and c is a color vector specified by a triplet of hues {R,G,B} (red, green, and blue).

Signals

a *signal* is a function of time, $e.g.$,

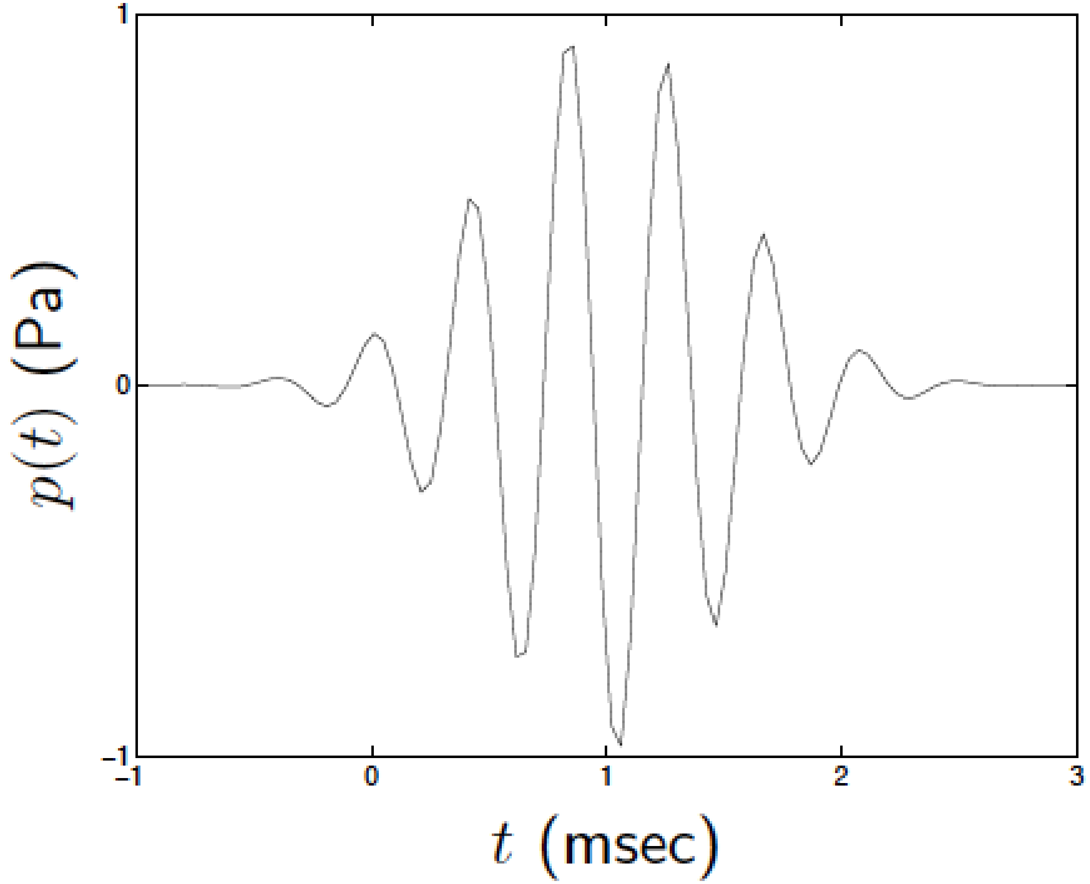
- f is the force on some mass
- v_{out} is the output voltage of some circuit
- p is the acoustic pressure at some point

notation:

- f, v_{out}, p or $f(\cdot)$, $v_{\text{out}}(\cdot)$, $p(\cdot)$ refer to the *whole signal* or *function*
- $f(t)$, $v_{\text{out}}(1.2)$, $p(t+2)$ refer to the *value* of the signals at times t, 1.2, and $t+2$, respectively

for *times* we usually use symbols like t, τ, t_1, . . .

Example



Domain of a signal

domain of a signal: t's for which it is defined

some common domains:

- all t, $i.e.$, **R**
- nonnegative t: $t \geq 0$
  (here $t = 0$ just means some starting time of interest)
- t in some interval: $a \leq t \leq b$
- t at uniformly sampled points: $t = kh + t_0$, $k = 0, \pm 1, \pm 2, \ldots$
- *discrete-time signals* are defined for integer t, $i.e.$, $t = 0, \pm 1, \pm 2, \ldots$
  (here t means sample time or epoch, not real time in seconds)

we'll usually study signals defined on all reals, or for nonnegative reals

Dimension & units of a signal

dimension or **type** of a signal u, $e.g.$,

- *real-valued* or *scalar signal*: $u(t)$ is a real number (scalar)
- *vector signal*: $u(t)$ is a vector of some dimension
- *binary signal*: $u(t)$ is either 0 or 1

we'll usually encounter *scalar signals*

example: a vector-valued signal

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

might give the voltage at three places on an antenna

physical units of a signal, $e.g.$, V, mA, $\mathrm{m/sec}$

sometimes the physical units are 1 ($i.e.$, unitless) or unspecified

Common signals with names

- a constant (or static or DC) signal: $u(t) = a$, where a is some constant
- the *unit step* signal (sometimes denoted $1(t)$ or $U(t)$),

$$u(t) = 0 \text{ for } t < 0, \quad u(t) = 1 \text{ for } t \geq 0$$

- the *unit ramp* signal,

$$u(t) = 0 \text{ for } t < 0, \quad u(t) = t \text{ for } t \geq 0$$

- a *rectangular pulse* signal,

$$u(t) = 1 \text{ for } a \leq t \leq b, \quad u(t) = 0 \text{ otherwise}$$

- a *sinusoidal* signal:

$$u(t) = a\cos(\omega t + \phi)$$

a, b, ω, ϕ are called signal *parameters*

Real signals

most real signals, *e.g.*,

- AM radio signal
- FM radio signal
- cable TV signal
- audio signal
- NTSC video signal (National Television System Committee)
- 10BT ethernet signal
- telephone signal

aren't given by mathematical formulas, but they do have defining characteristics

Measuring the size of a signal

size of a signal u is measured in many ways

for example, if $u(t)$ is defined for $t \geq 0$:

- *integral square* (or *total energy*): $\int_0^\infty u(t)^2 \, dt$
- squareroot of total energy
- *integral-absolute value*: $\int_0^\infty |u(t)| \, dt$
- *peak* or *maximum absolute value* of a signal: $\max_{t \geq 0} |u(t)|$
- *root-mean-square* (RMS) value: $\left(\lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)^2 \, dt \right)^{1/2}$
- *average-absolute* (AA) value: $\lim_{T \to \infty} \frac{1}{T} \int_0^T |u(t)| \, dt$

for some signals these measures can be infinite, or undefined

example: for a sinusoid $u(t) = a\cos(\omega t + \phi)$ for $t \geq 0$

- the peak is $|a|$
- the RMS value is $|a|/\sqrt{2} \approx 0.707|a|$
- the AA value is $|a|2/\pi \approx 0.636|a|$
- the integral square and integral absolute values are ∞

the *deviation* between two signals u and v can be found as the size of the difference, *e.g.*, $\mathrm{RMS}(u - v)$

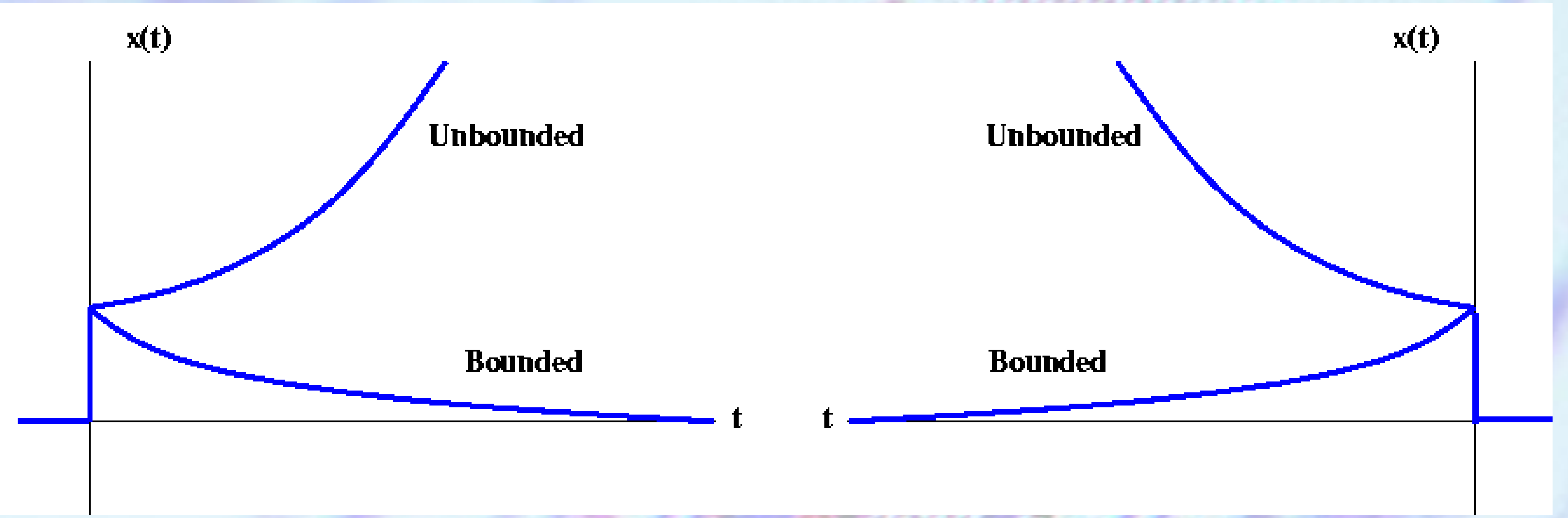
Qualitative properties of signals

- u *decays* if $u(t) \to 0$ as $t \to \infty$
- u *converges* if $u(t) \to a$ as $t \to \infty$ (a is some constant)
- u is *bounded* if its peak is finite
- u is *unbounded* or *blows up* if its peak is infinite
- u is *periodic* if for some $T > 0$, $u(t+T) = u(t)$ holds for all t

in practice we are interested in more specific quantitative questions, $e.g.$,

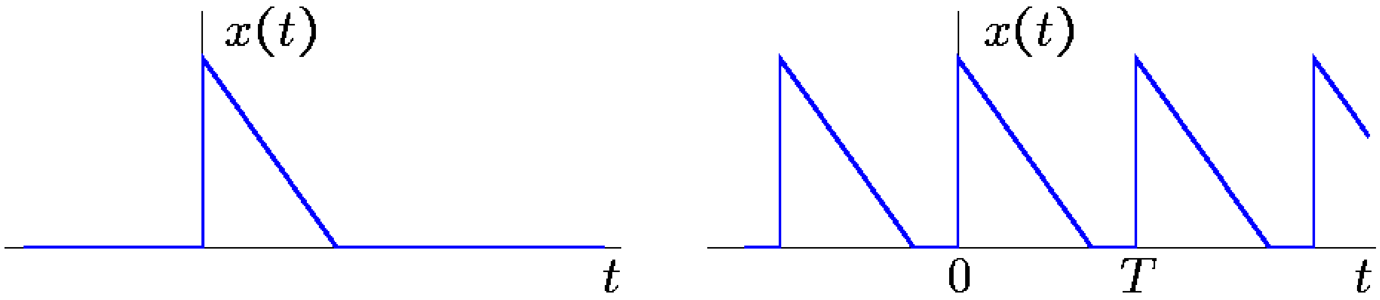
- how fast does u decay or converge?
- how large is the peak of u?

Bounded and Unbounded Signals



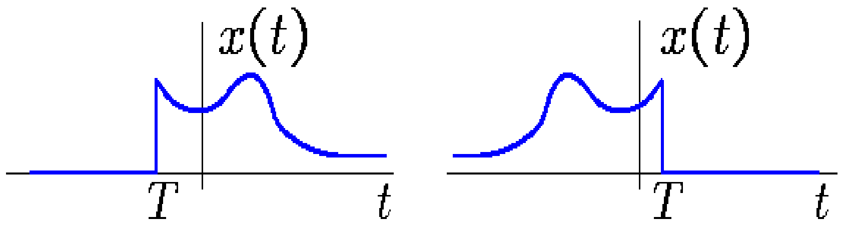
Periodic and A-periodic Signals

Periodic signals are such that $x(t+T) = x(t)$ for all t. The smallest value of T that satisfies the definition is called the *period*. Below on the left below is an aperiodic signal, with a periodic signal shown on the right.



Right- and Left-Sided Signals

A right-sided signal is zero for $t < T$ and a left-sided signal is zero for $t > T$ where T can be positive or negative.



Week 2

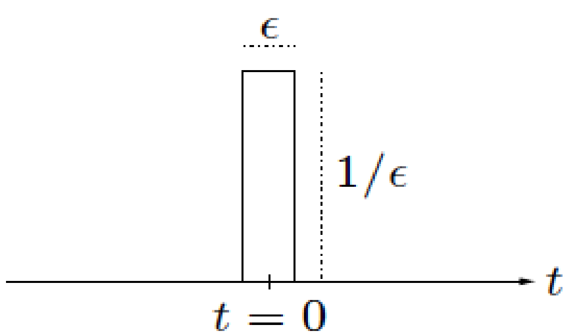
Slide 36-50

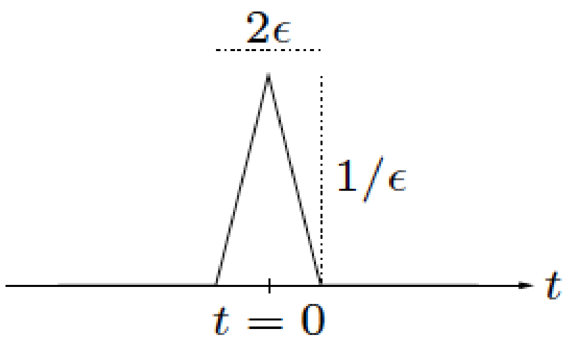
Impulsive signals

(Dirac's) **delta function** or **impulse** δ is an *idealization* of a signal that

- is very large near $t = 0$
- is very small away from $t = 0$
- has integral 1

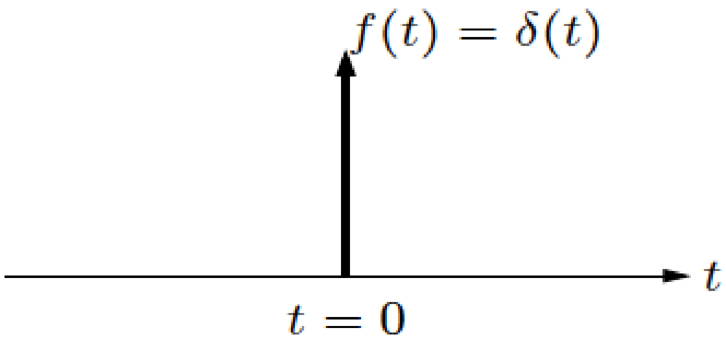
for example:

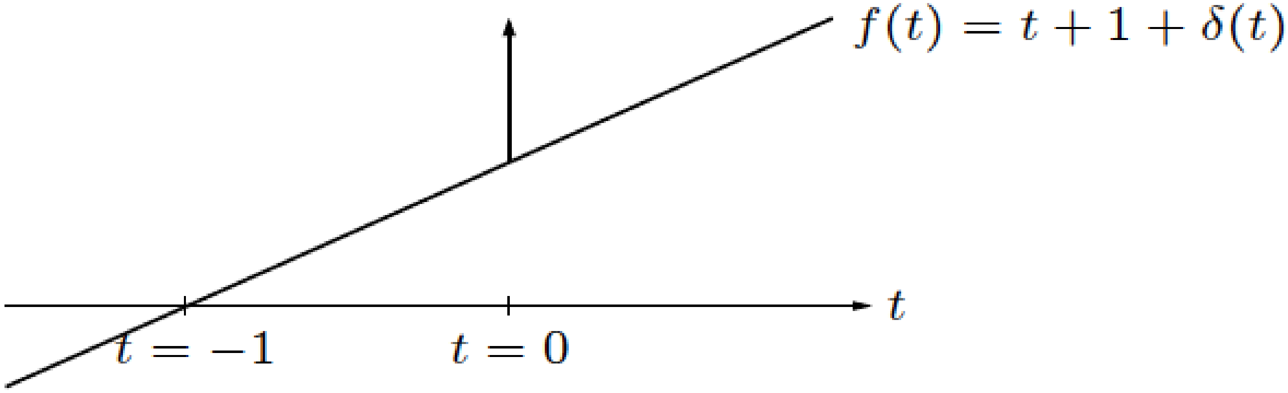




- the exact shape of the function doesn't matter
- ϵ is small (which depends on context)

on plots δ is shown as a solid arrow:





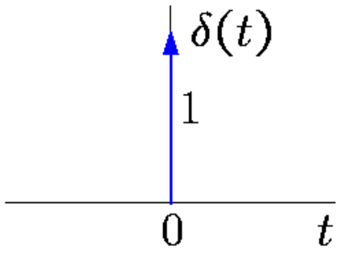
Unit Impulse Function

The unit impulse $\delta(t)$, aka the Dirac delta function, is not a function in the ordinary sense. It is defined by the integral relation

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0),$$

and is called a *generalized function*. The unit impulse is not defined in terms of its values, but is defined by how it acts inside an integral when multiplied by a smooth function $f(t)$. To see that the area of the unit impulse is 1, choose $f(t) = 1$ in the definition. We represent the unit impulse schematically as shown below; the number next to the impulse is its area.

Unit impulse



Formal properties

formally we **define** δ by the property that

$$\int_a^b f(t)\delta(t)\, dt = f(0)$$

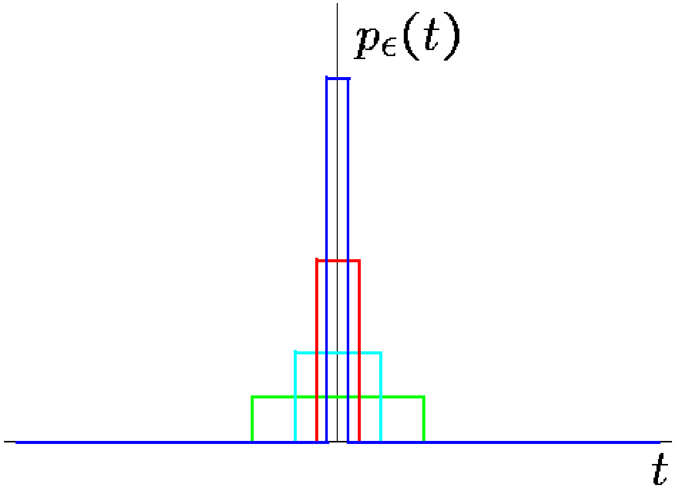
provided $a < 0$, $b > 0$, and f is continuous at $t = 0$

idea: δ acts over a time interval very small, over which $f(t) \approx f(0)$

- $\delta(t) = 0$ for $t \neq 0$
- $\delta(0)$ isn't really defined
- $\int_a^b \delta(t)\, dt = 1$ if $a < 0$ and $b > 0$
- $\int_a^b \delta(t)\, dt = 0$ if $a > 0$ or $b < 0$

Narrow Pulse Approximation

To obtain an intuitive feeling for the unit impulse, it is often helpful to imagine a set of rectangular pulses where each pulse has width ϵ and height $1/\epsilon$ so that its area is 1.



The unit impulse is the quintessential tall and narrow pulse!

Uses of the Unit Impulse

The unit impulse is a valuable idealization and is used widely in science and engineering. Impulses in time are useful idealizations.

- Impulse of current in time delivers a unit charge instantaneously to a network.
- Impulse of force in time delivers an instantaneous momentum to a mechanical system.

Physical interpretation

impulse functions are used to model physical signals

- that act over short time intervals
- whose effect depends on integral of signal

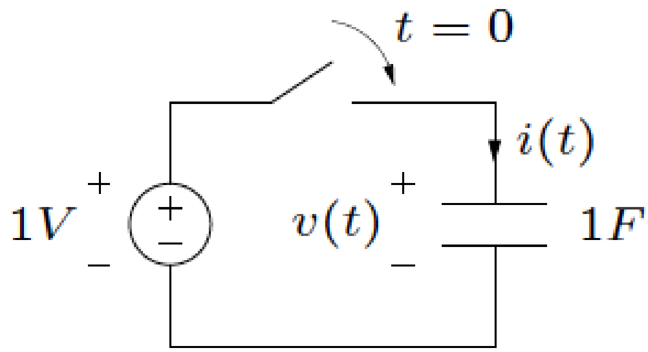
example: hammer blow, or bat hitting ball, at $t = 2$

- force f acts on mass m between $t = 1.999$ sec and $t = 2.001$ sec
- $\int_{1.999}^{2.001} f(t)\, dt = I$ (mechanical impulse, $\mathrm{N} \cdot \mathrm{sec}$)
- blow induces change in velocity of

$$v(2.001) - v(1.999) = \frac{1}{m} \int_{1.999}^{2.001} f(\tau)\, d\tau = I/m$$

for (most) applications we can model force as an impulse, at $t = 2$, with magnitude I

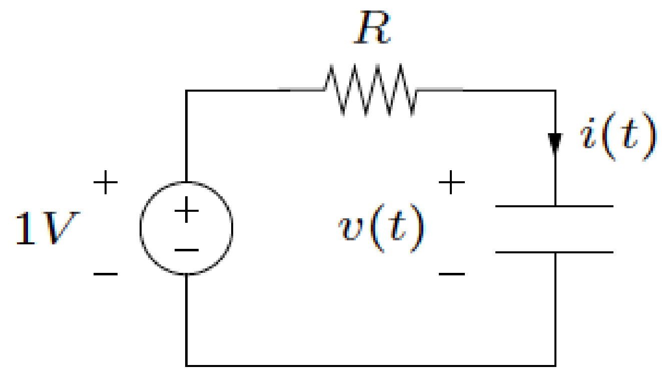
example: rapid charging of capacitor



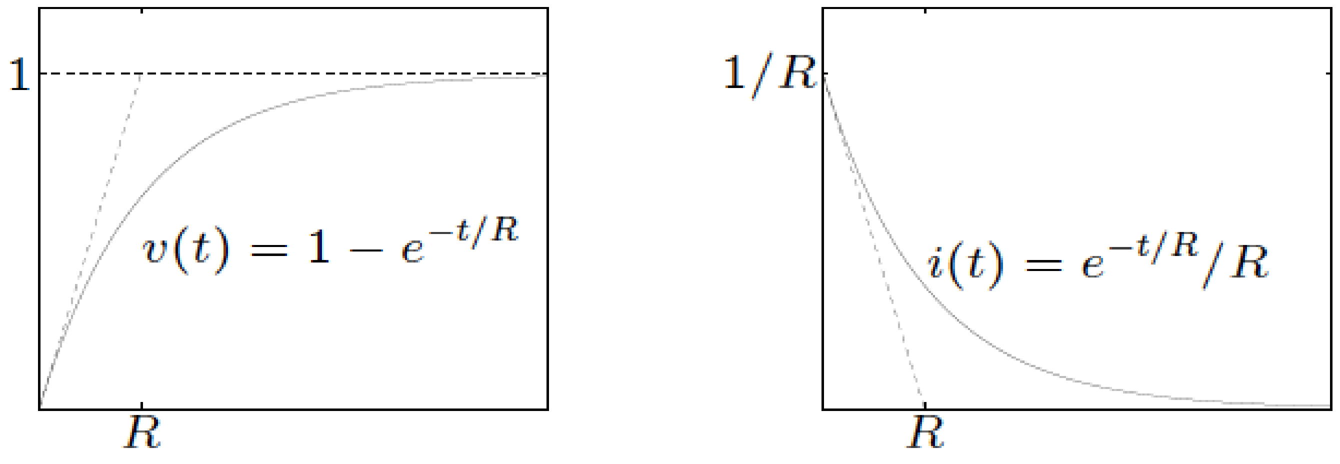
assuming $v(0) = 0$, what is $v(t)$, $i(t)$ for $t > 0$?

- $i(t)$ is very large, for a very short time
- a unit charge is transferred to the capacitor 'almost instantaneously'
- $v(t)$ increases to $v(t) = 1$ 'almost instantaneously'

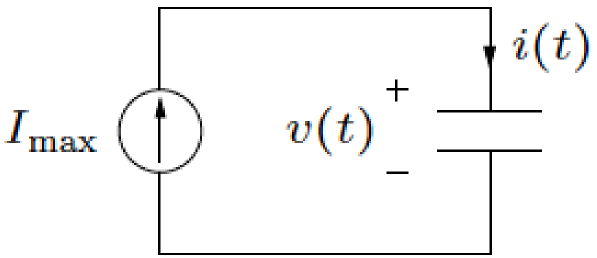
to calculate i, v, we need a more detailed model for example, include small resistance



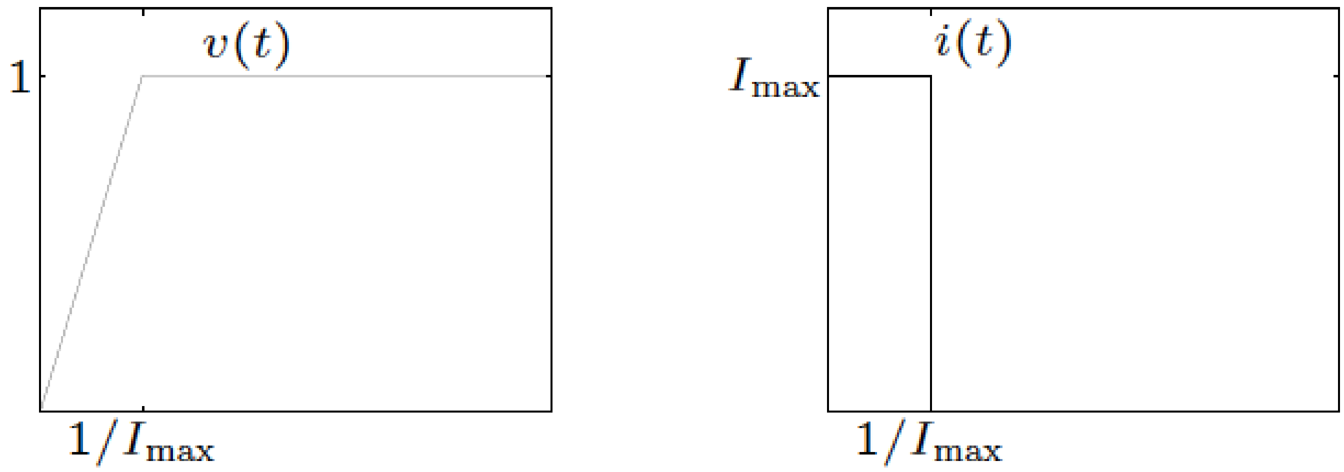
$$i(t) = \frac{dv(t)}{dt} = \frac{1 - v(t)}{R}, \quad v(0) = 0$$



as $R \to 0$, i approaches an impulse, v approaches a unit step as another example, assume the current delivered by the source is limited: if $v(t) < 1$, the source acts as a current source $i(t) = I_{\max}$



$$i(t) = \frac{dv(t)}{dt} = I_{\max}, \quad v(0) = 0$$



as $I_{\max} \to \infty$, i approaches an impulse, v approaches a unit step in conclusion,

- large current i acts over very short time between $t = 0$ and ϵ
- total charge transfer is $\int_0^\epsilon i(t)\,dt = 1$
- resulting change in $v(t)$ is $v(\epsilon) - v(0) = 1$
- can approximate i as impulse at $t = 0$ with magnitude 1

modeling current as impulse

- obscures details of current signal
- obscures details of voltage change during the rapid charging
- preserves total change in charge, voltage
- is reasonable model for time scales $\gg \epsilon$

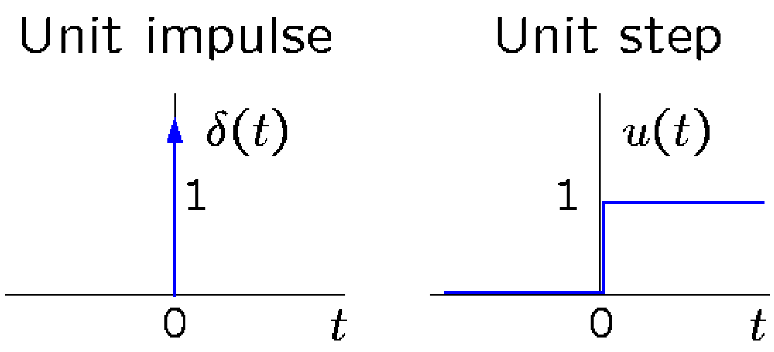
Unit Step Function

Integration of the unit impulse yields the unit step function

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau,$$

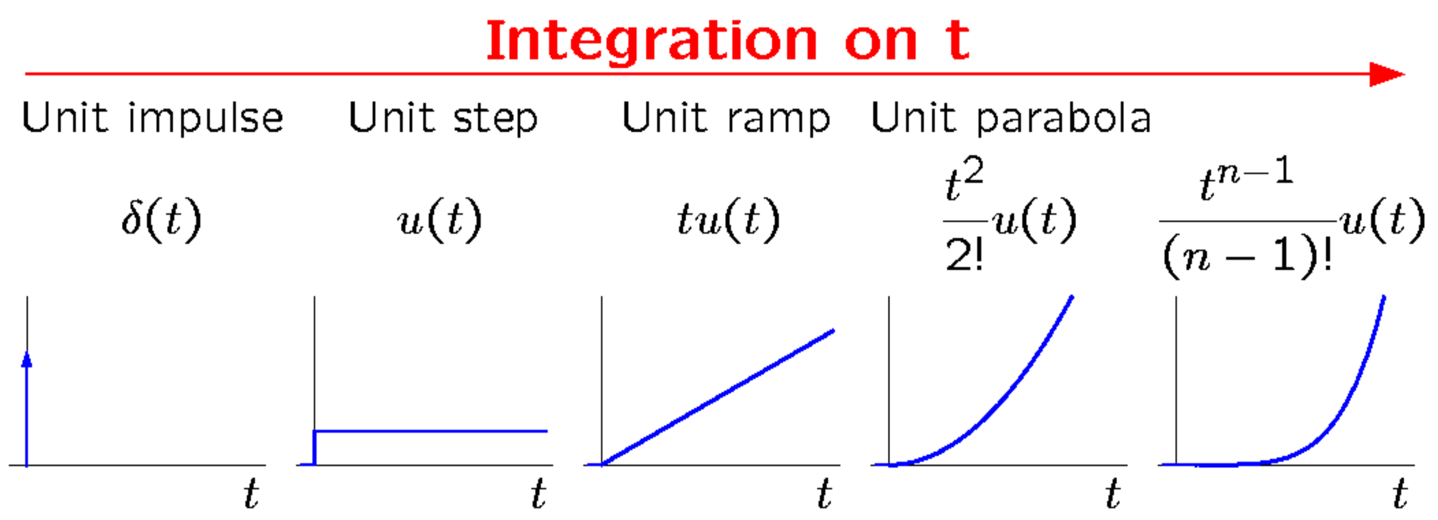
which is defined as

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$

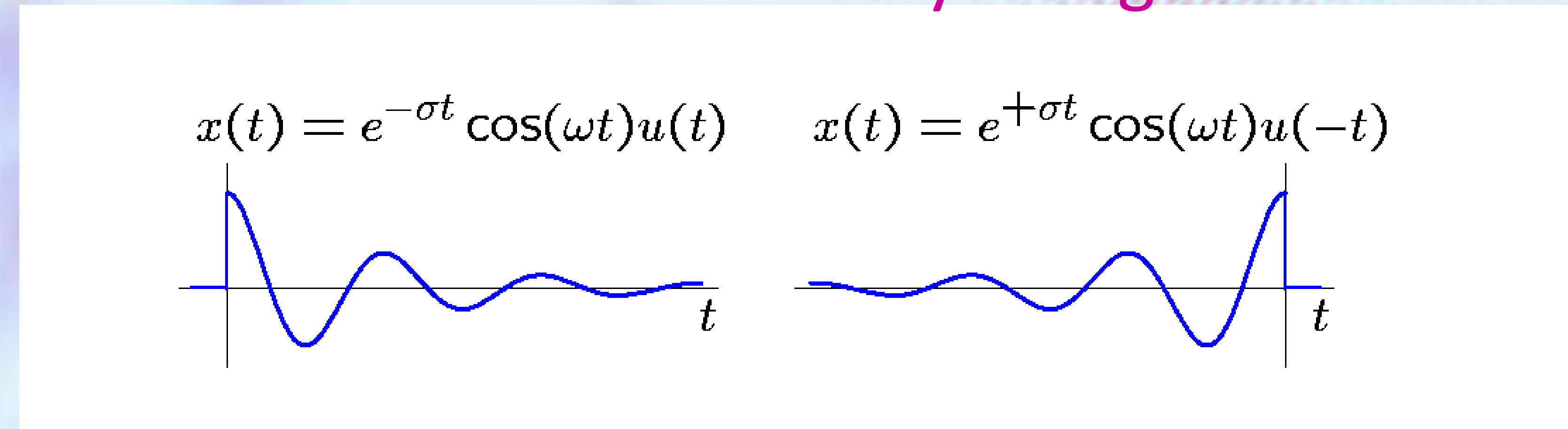


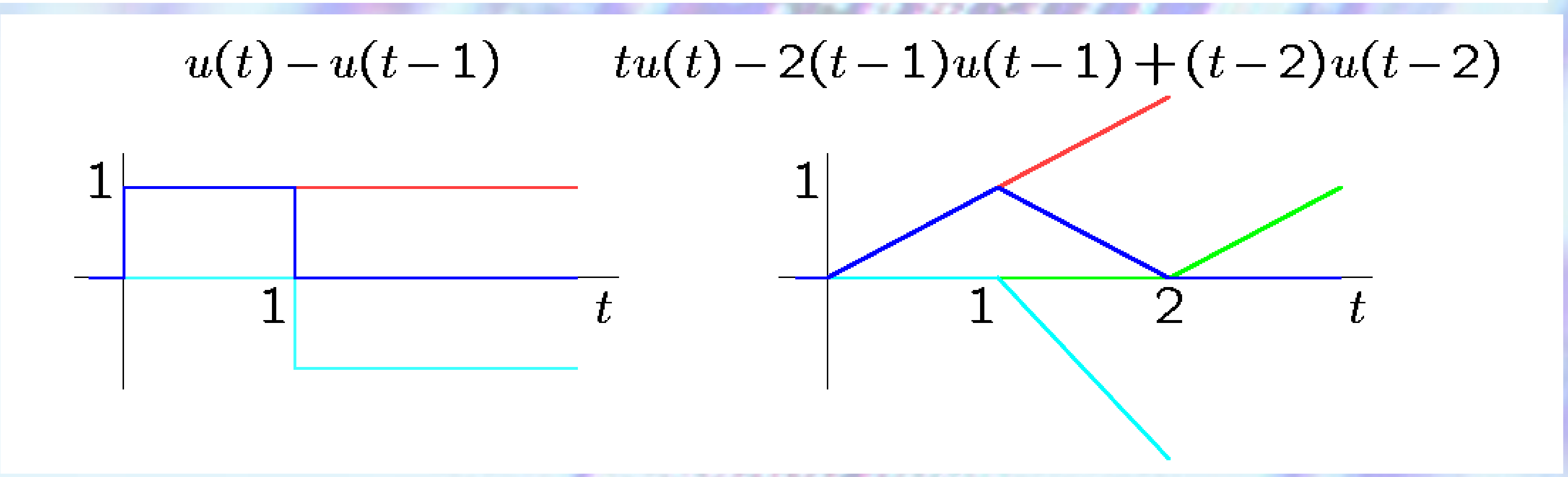
Successive Integrations of the Unit Impulse Function

Successive integration of the unit impulse yields a family of functions.



Building Block Signals can be used to create a rich variety of Signals





Prepared by Noor Md Shahriar

Conclusions

- We are awash in a sea of signals.
- Signal categories — identity of independent variable, dimensionality, CT or DT, real or complex, periodic or aperiodic, causality, bounded, even & odd, etc.
- Building block signals — eternal complex exponentials and singularity functions — are a rich class of signals and we will show that they can be summed to represent virtually any signal of physical interest.

Week 3

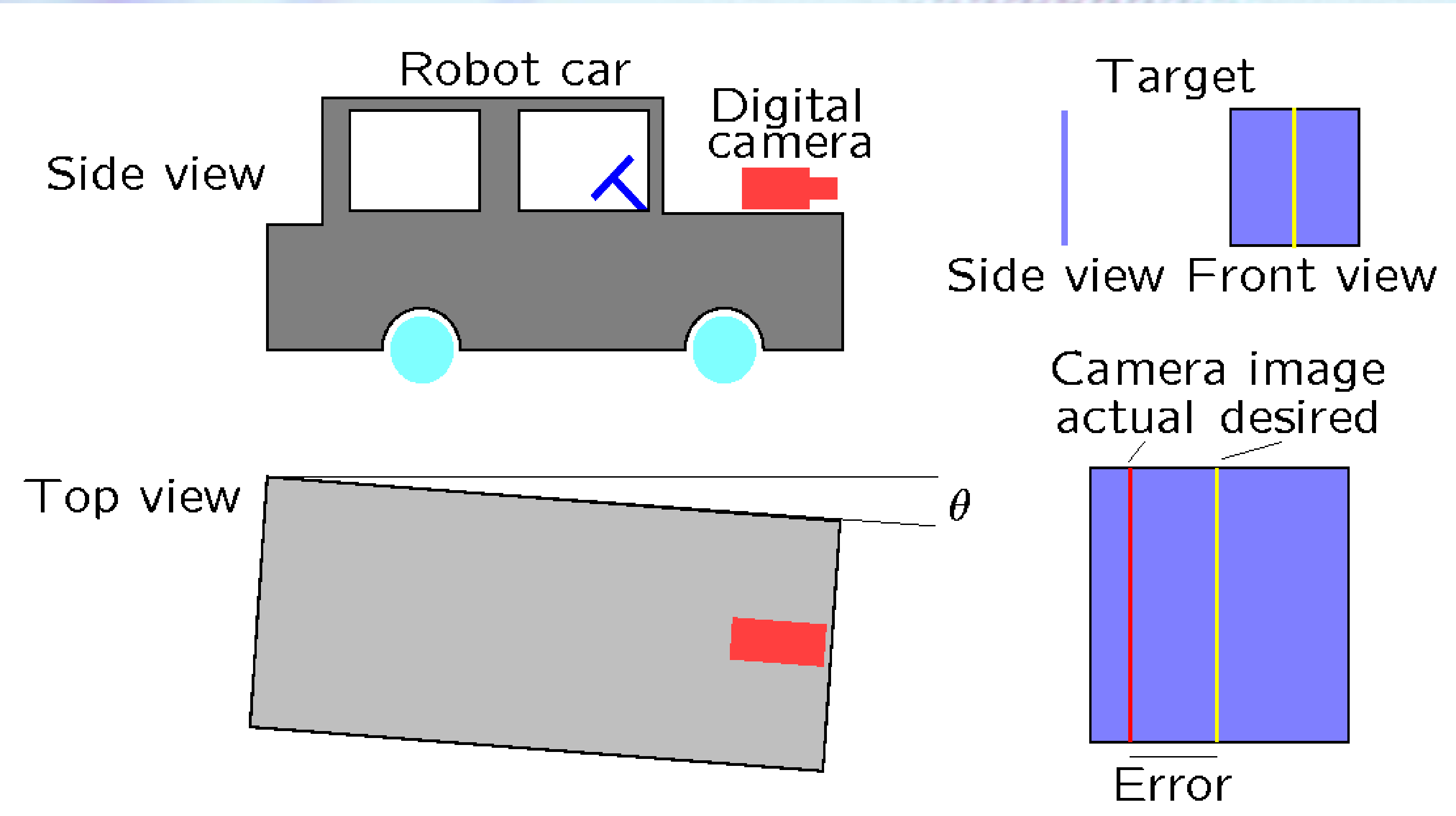
Slide 51-66

Outline - Systems

- How do we construct complex systems
  - Using Hierarchy
  - Composing simpler elements
- System Representations
  - Physical, differential/difference Equations, etc.
- System Properties
  - Causality, Linearity and Time-Invariance

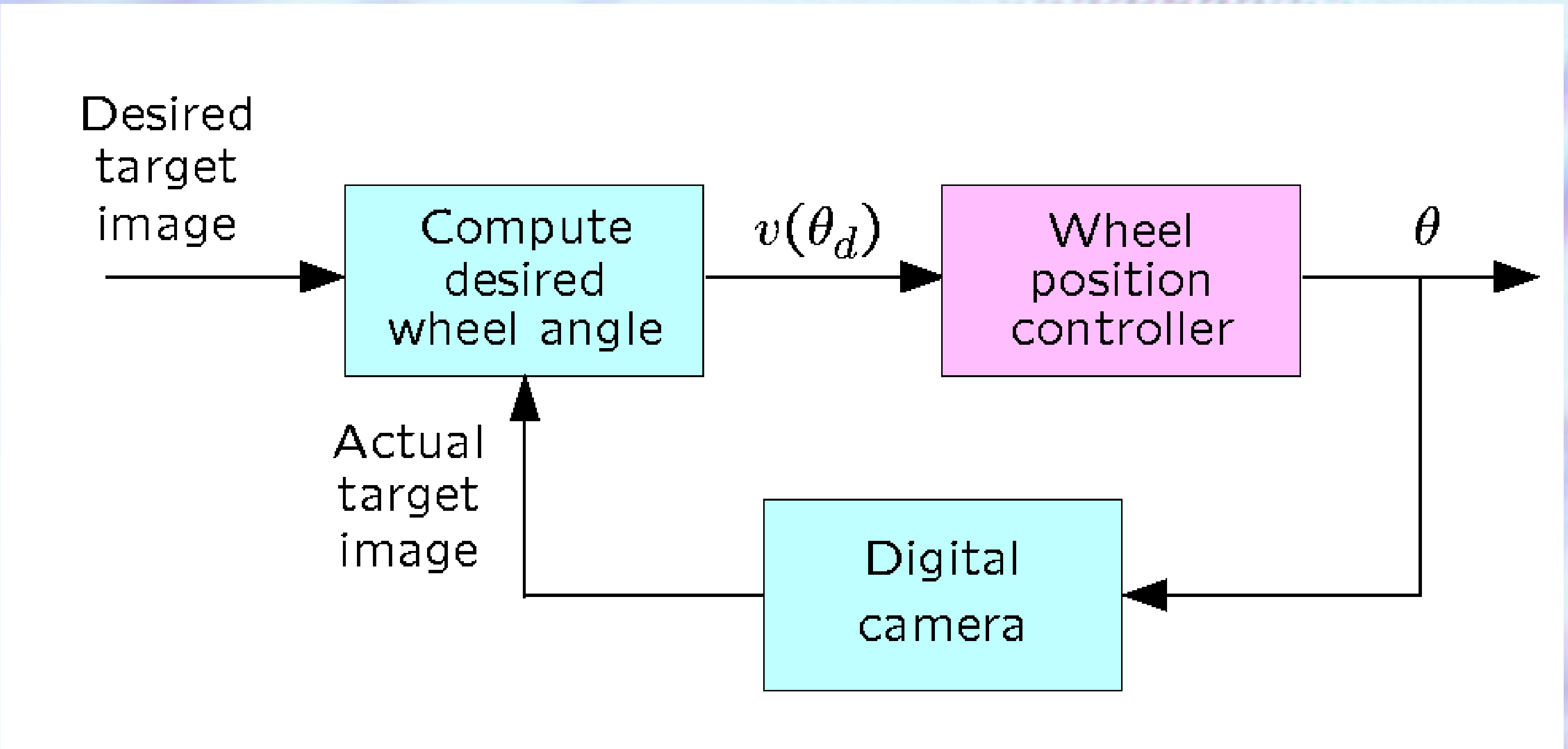
Hierarchical Design

Robot Car



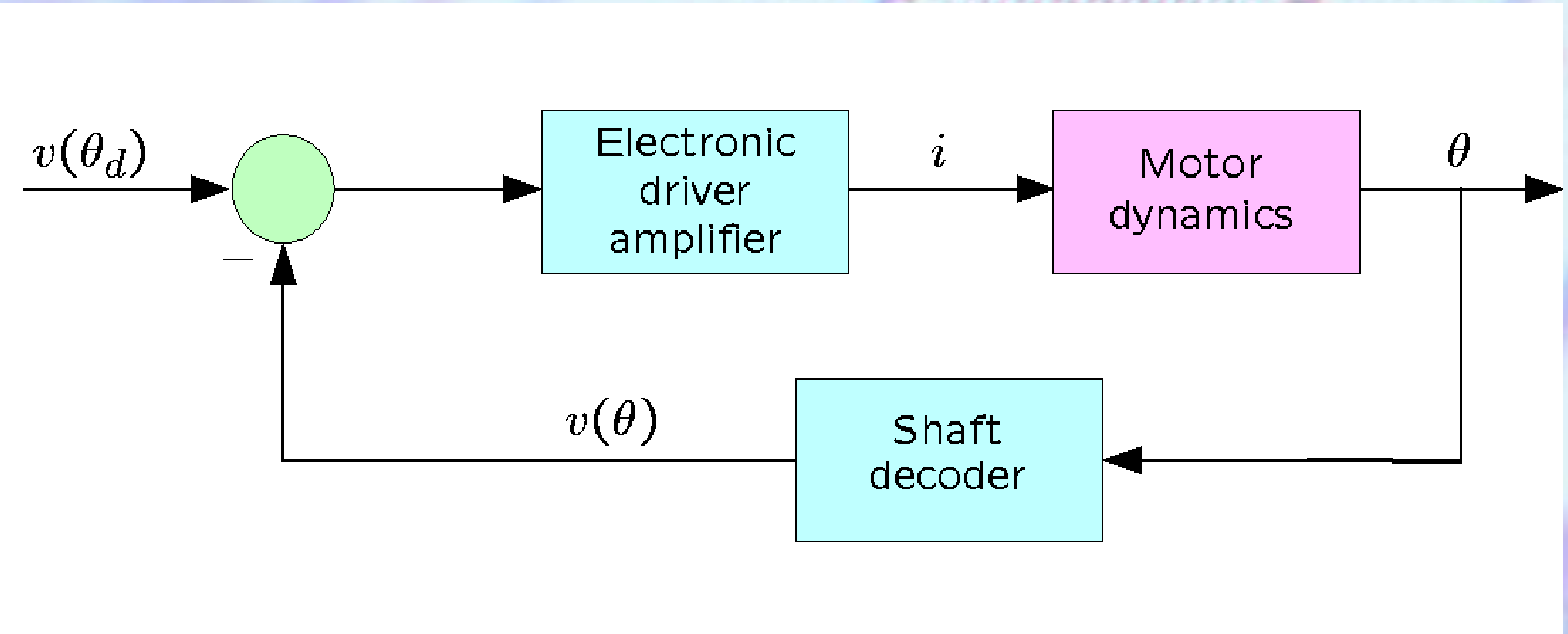
Robot Car Block Diagram

Top Level of Abstraction



Wheel Position Controller Block Diagram

2nd Level of the Hierarchy



Motor Dynamics Differential Equations

3nd Level of the Hierarchy

- Motor Current $i(t)$,
- Angular Displacement, $\theta(t)$,
- Constants: Friction, B, Moment of Inertia J, torque conversion k.

The torque balance differential equation:

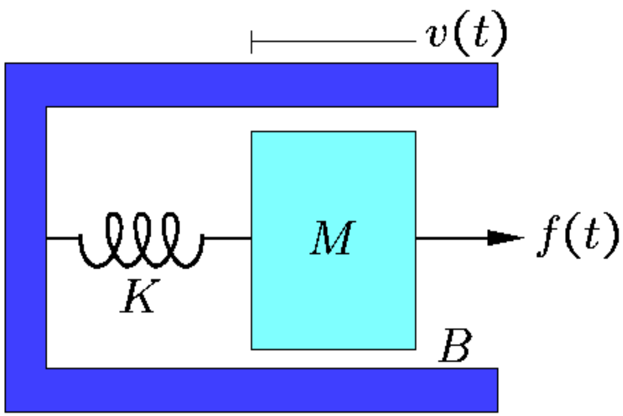
$$\underbrace{ki(t)}_{\text{electric origin}} \quad \underbrace{-B\frac{d\theta(t)}{dt}}_{\text{friction}} = \underbrace{J\frac{d^2\theta(t)}{dt^2}}_{\text{inertia}}$$

Observations

- If we "flatten" the hierarchy, the system becomes very complex
- Human designed systems are often created hierarchically.
- Block input/output relations provide communication mechanisms for team projects

Compositional Design

Mechanics - Sum Element Forces

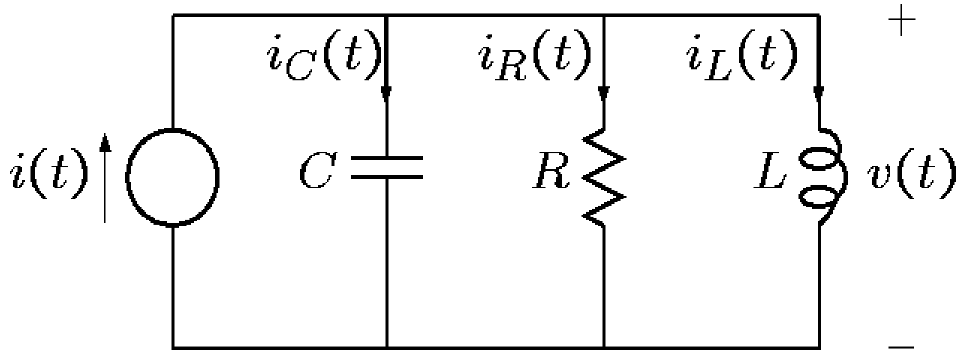


- M = mass, $v(t)$ = velocity
- B = friction, K = spring
- $f(t)$ = external force.

Summing forces yields

$$f(t) = \underbrace{M\frac{dv(t)}{dt}}_{\text{inertial force}} + \underbrace{Bv(t)}_{\text{friction force}} + \underbrace{K\int_{-\infty}^{t} v(\tau)\,d\tau}_{\text{spring force}}.$$

Circuit - Sum Element Currents



Summing currents yields

$$i(t) = \underbrace{C\frac{dv(t)}{dt}}_{\text{capacitance}} + \underbrace{\frac{v(t)}{R}}_{\text{resistance}} + \underbrace{\frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau}_{\text{inductance}} .$$

System Representation

Differential Equation representation

- Mechanical and Electrical Systems Dynamically Analogous

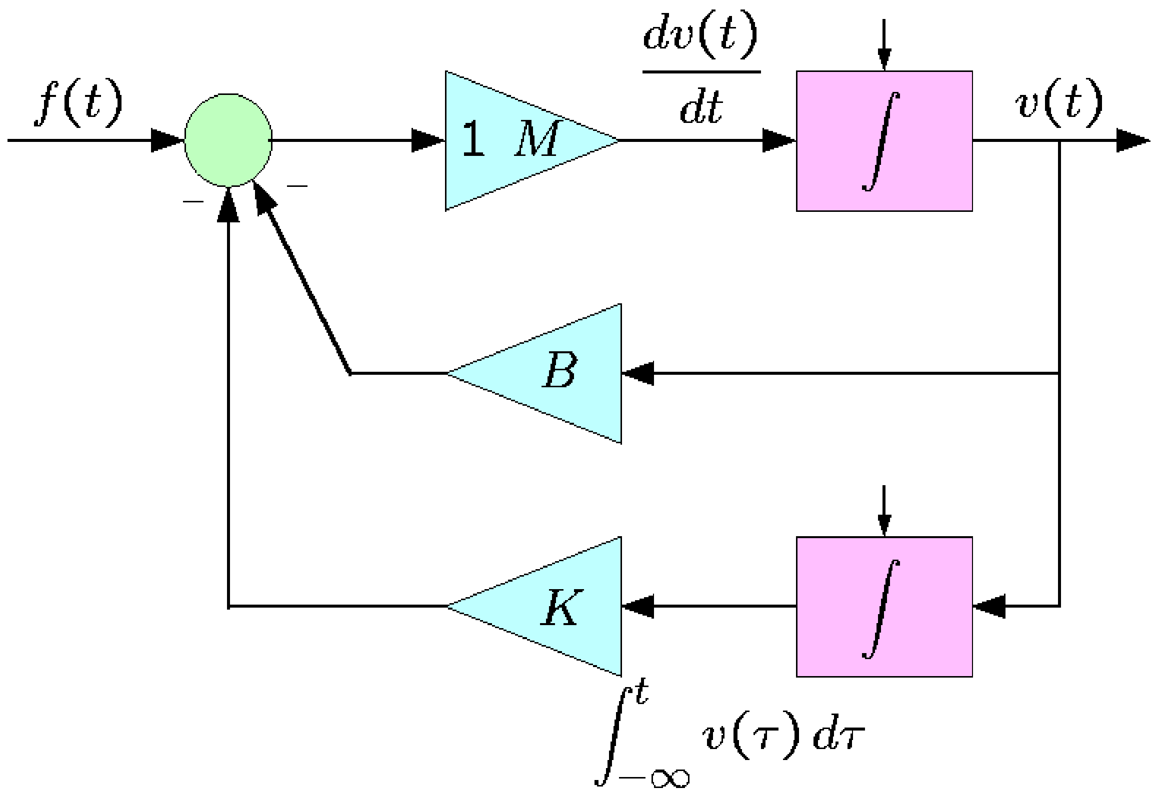
$$f(t) = M\frac{dv(t)}{dt} + Bv(t) + K\int_{-\infty}^{t} v(\tau)\,d\tau,$$

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau.$$

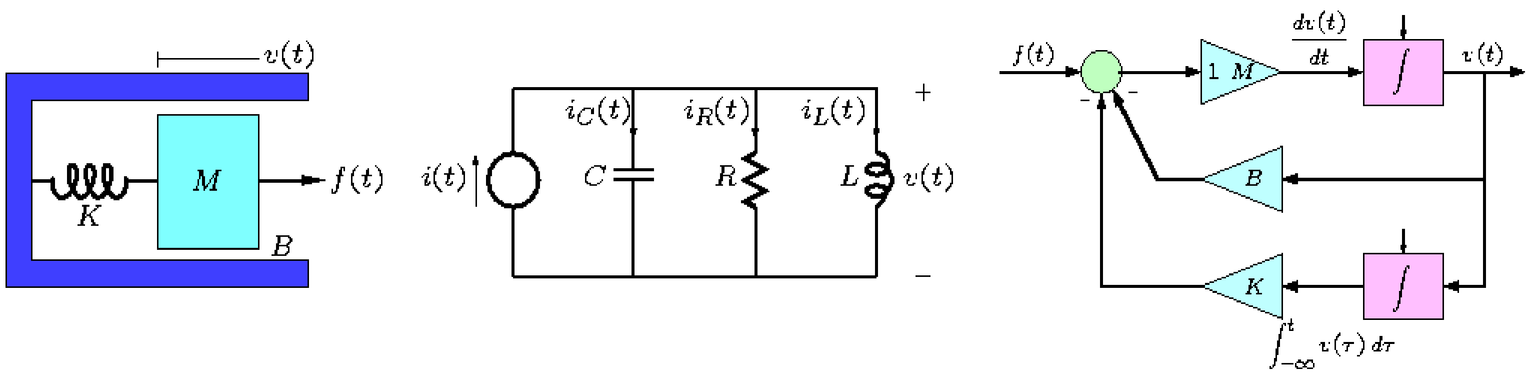
- Can reason about the system using either physical representation.

Integrator-Adder-Gain Block Diagram

$$f(t) = M\frac{dv(t)}{dt} + Bv(t) + K\int_{-\infty}^{t} v(\tau)\,d\tau.$$



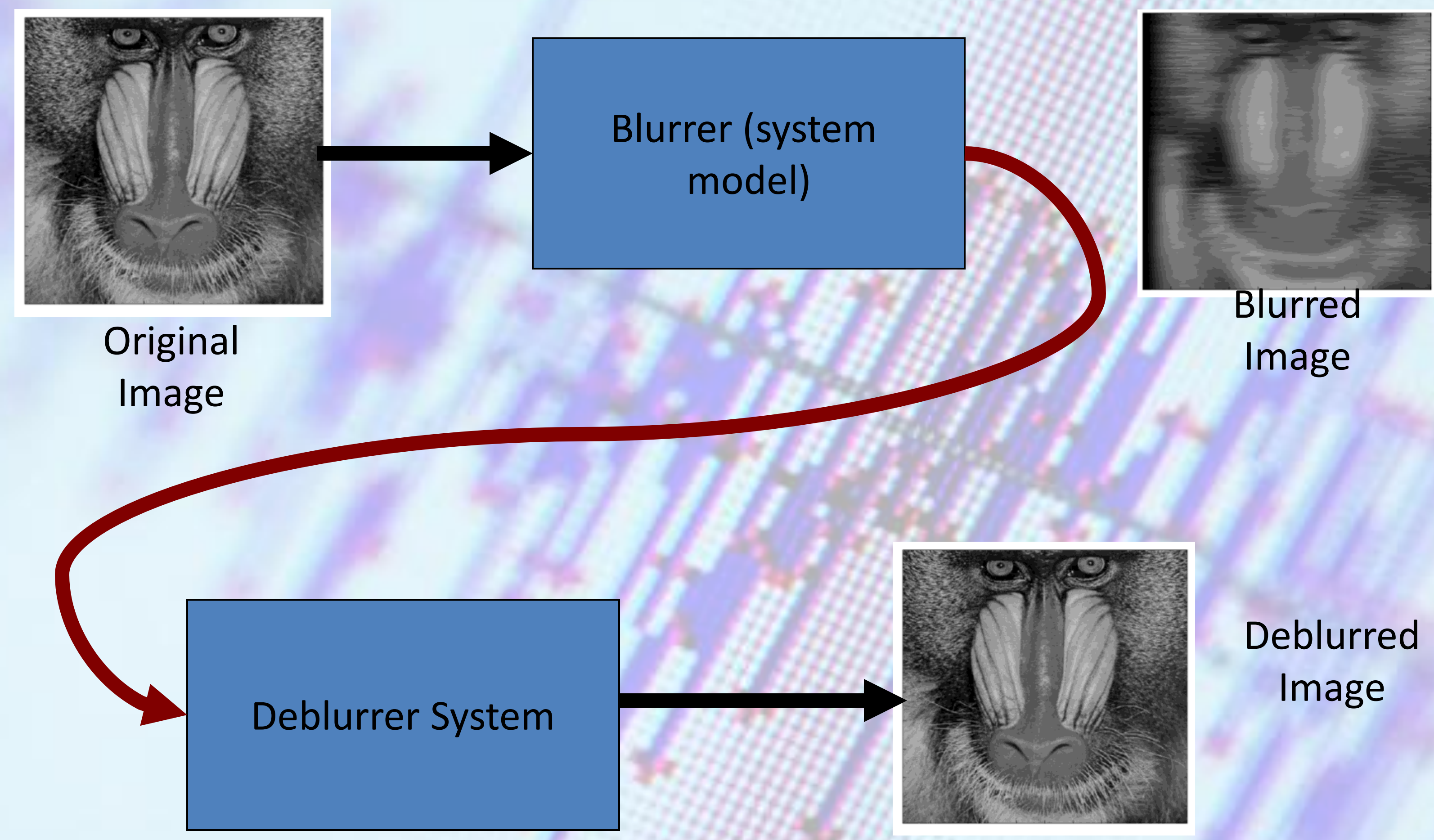
Four Representations for the same dynamic behavior



$$f(t) = M\frac{dv(t)}{dt} + Bv(t) + K\int_{-\infty}^{t} v(\tau)\,d\tau.$$

Pick the representation that makes it easiest to solve the problem

Discrete-Time Example - Blurred Mandril



Difference Equation Representation

- Difference Equation Representation of the model of a Blurring System

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] = x[n]$$

- Deblurring System

$$z[n] = b_0 y[n] + b_1 y[n-1] + b_2 y[n-2] + b_3 y[n-3]$$

How do we get $z[n] \approx x[n]$

- The *difference equation* is a formula for computing an output sample at time based on past and present input samples and past output samples in the time domain:

$$y(n) = b_0\,x(n) + b_1\,x(n-1) + \cdots + b_M\,x(n-M) - a_1\,y(n-1) - \cdots - a_N\,y(n-N)$$

$$= \sum_{i=0}^{M} b_i\,x(n-i) - \sum_{j=1}^{N} a_j\,y(n-j)$$

where x is the input signal, y is the output signal, and the constants $b_i, i = 0, 1, 2, \ldots, M$, $a_i, i = 1, 2, \ldots, N$ are called the *coefficients*

As a specific example, the difference equation

$$y(n) = 0.01\,x(n) + 0.002\,x(n-1) + 0.99\,y(n-1)$$

specifies a digital filtering operation, and the coefficient sets $(0.01, 0.002)$ and (0.99) fully characterize the filter. In this example, we have $M = N = 1$

Observations

- CT System representations include circuit and mechanical analogies, differential equations, and Integrator-Adder-Gain block diagram.
- Discrete-Time Systems can be represented by difference equations.
- The Difference Equation representation does not help us design the mandril deblurring
- New representations and tools for manipulating are needed!

Prepared by Noor Md Shahriar

Week 4

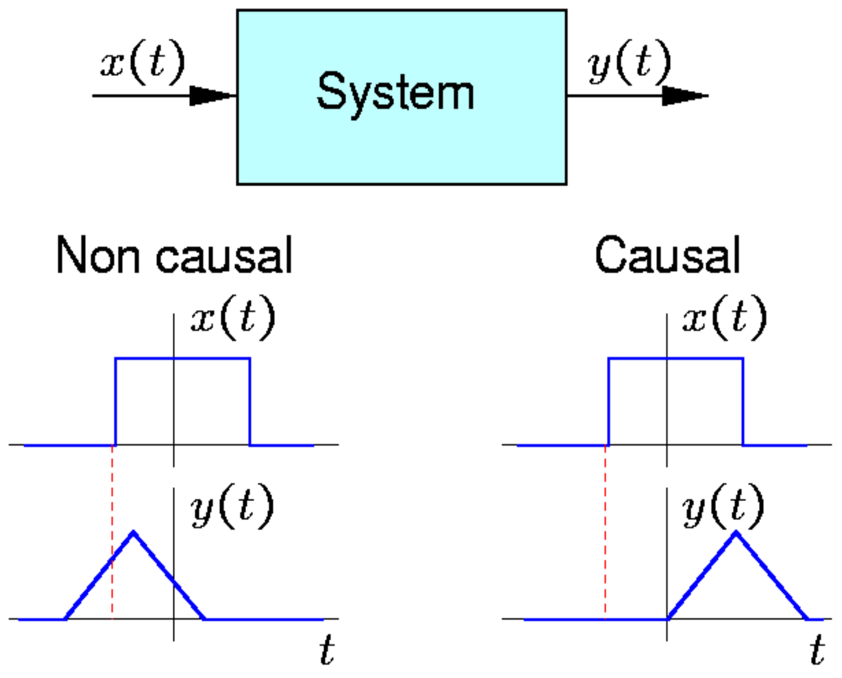
Slide 68-83

System Properties

- Important practical/physical implications
- Help us select appropriate representations
- They provide us with insight and structure that we can exploit both to analyze and understand systems more deeply.

Causal and Non-causal Systems

For a causal system the output at time t_o depends only on the input for $t \leq t_o$, i.e., the system cannot anticipate the input.



Observations on Causality

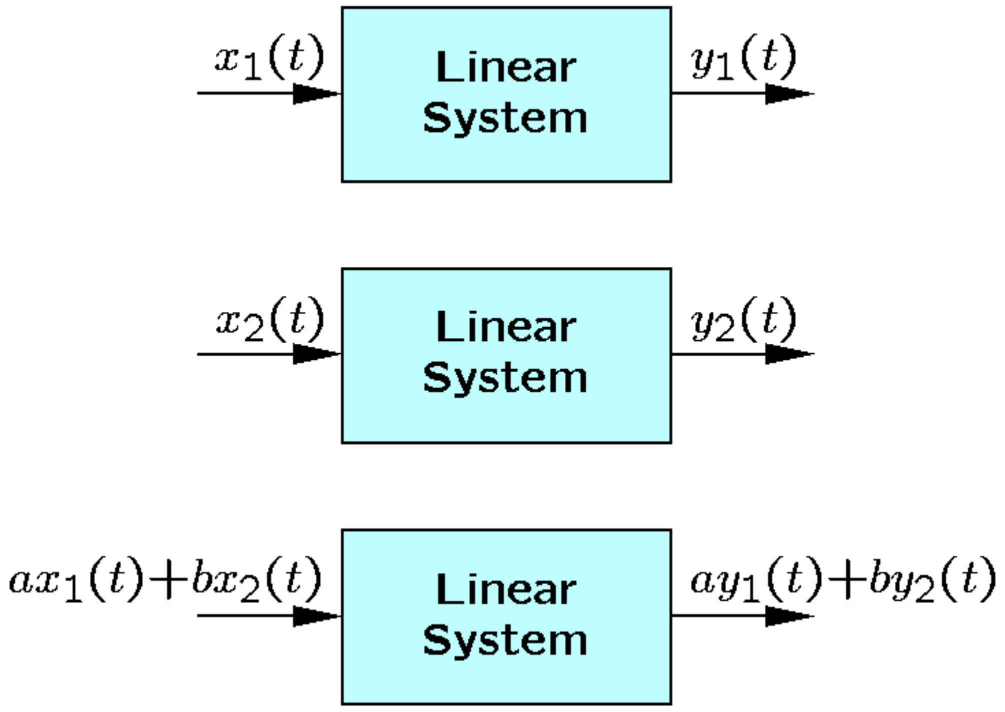
- A system is causal if the output does <u>not</u> anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.

- <u>All</u> real-time physical systems are <u>causal</u>, because time only moves forward. Effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)

- Causality does <u>not</u> apply to spatially varying signals. (We can move both left and right, up and down.)

- Causality does <u>not</u> apply to systems processing <u>recorded</u> signals, e.g. taped sports games vs. live broadcast.

Linearity



for all $x_1(t)$, $x_2(t)$, a, and b.

Key Property of Linear Systems

- Superposition

If $$x_k[n] \rightarrow y_k[n]$$

Then $$\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$$

Linearity and Causality

- A linear system is causal if and only if it satisfies the conditions of <u>initial rest</u>:

$$x(t) = 0 \text{ for } t \leq t_0 \rightarrow y(t) = 0 \text{ for } t \leq t_0 \quad (*).$$

Time-Invariance

- Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for <u>any</u> input $x[n]$ and <u>any</u> time shift n_0,

  If $\quad x[n] \rightarrow y[n]$

  then $\quad x[n - n_0] \rightarrow y[n - n_0]$.

- Similarly for CT time-invariant system,

  If $\quad x(t) \rightarrow y(t)$

  then $\quad x(t - t_o) \rightarrow y(t - t_o)$.

Interesting Observation

Fact: If the input to a TI System is periodic, then the output is periodic with the same period.

"Proof": Suppose $x(t + T) = x(t)$

and $x(t) \rightarrow y(t)$

Then by TI

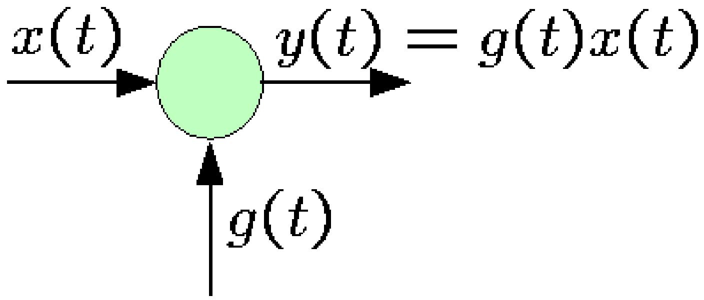
$x(t + T) \rightarrow y(t + T)$.

↑ ↑

These are the same input!

So these must be the same output, i.e., $y(t) = y(t + T)$.

Example - Multiplier



- Is this system linear?

- Is this system time–invariant?

Multiplier Linearity

Let

$$y_1(t) = g(t)x_1(t) \text{ and } y_2(t) = g(t)x_2(t).$$

By definition the response to

$$x(t) = ax_1(t) + bx_2(t),$$

is

$$y(t) = g(t)(ax_1(t) + bx_2(t)).$$

This can be rewritten as

$$\begin{aligned} y(t) &= ag(t)x_1(t) + bg(t)x_2(t) \\ y(t) &= ay_1(t) + by_2(t). \end{aligned}$$

Therefore, the system is linear.

Multiplier – Time Varying

Now suppose that $x_1(t) = x(t)$ and $x_2(t) = x(t - \tau)$, and the response to these two inputs are $y_1(t)$ and $y_2(t)$, respectively. Note that

$$y_1(t) = y(t) = g(t)x(t),$$

and

$$y_2(t) = g(t)x(t - \tau) \neq y(t - \tau).$$

Therefore, the system is time-varying.

Example – Constant Addition

Suppose the relation between the output $y(t)$ and input $x(t)$ is given $y(t) = x(t) + K$, where K is some constant. Is this system linear?

Solution — Addition of a constant

Note, that if the input is $x_1(t) + x_2(t)$ then the output will be

$$y(t) = x_1(t) + x_2(t) + K \neq y_1(t) + y_2(t) = (x_1(t) + K) + (x_2(t) + K).$$

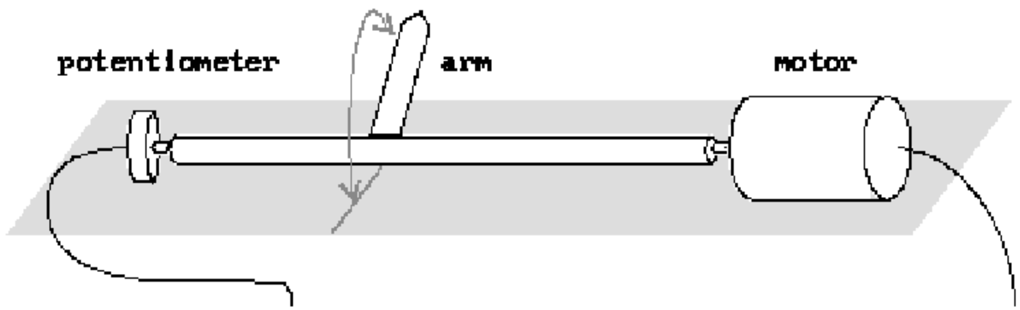
Therefore, this system is not linear.

In general, it can be shown that for a linear system if $x(t) = 0$ then $y(t) = 0$. Using the definition of linearity, choose $a = b = 1$ and $x_2 = -x_1(t)$ then $x(t) = x_1(t) + x_2(t) = 0$ and $y(t) = y_1(t) + y_2(t) = 0$.

Two-minute miniquiz problem

Problem 2-1

Consider the robot arm:



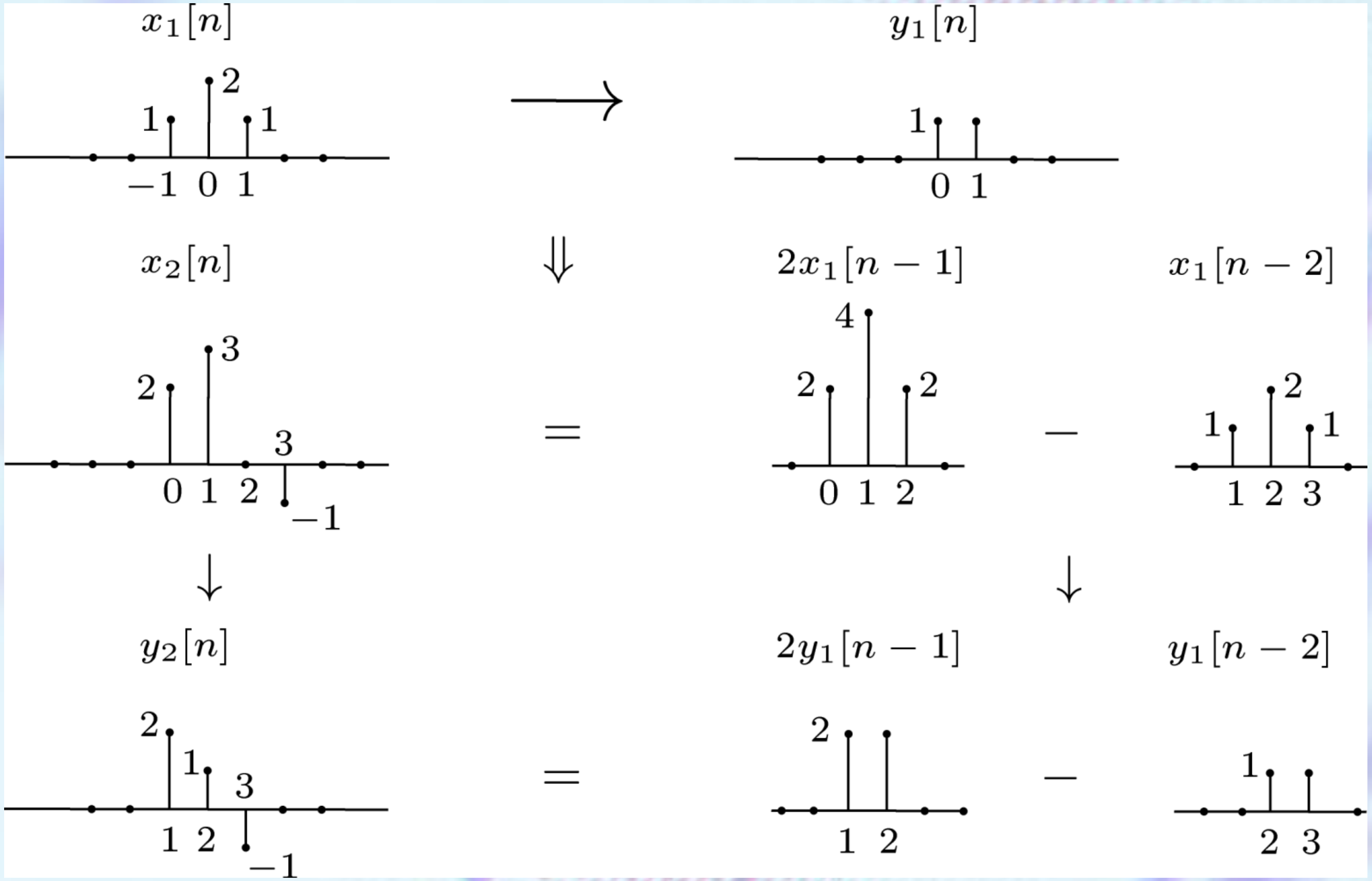
- How will you model the system?

- How would you control the position?

Linear Time-Invariant (LTI) Systems

- Focus of <u>most</u> of this course
  - Practical importance
  - The powerful analysis tools associated with LTI systems
- A basic fact: If we know the response of an LTI system to some inputs, we actually know the response to many inputs

Example – DT LTI System



Conclusions

- Systems are typically described by an arrangement of subsystems each of which is defined by a functional relation.

- Many different physical systems are defined by the same mathematical model so that understanding one system leads to an understanding of others.

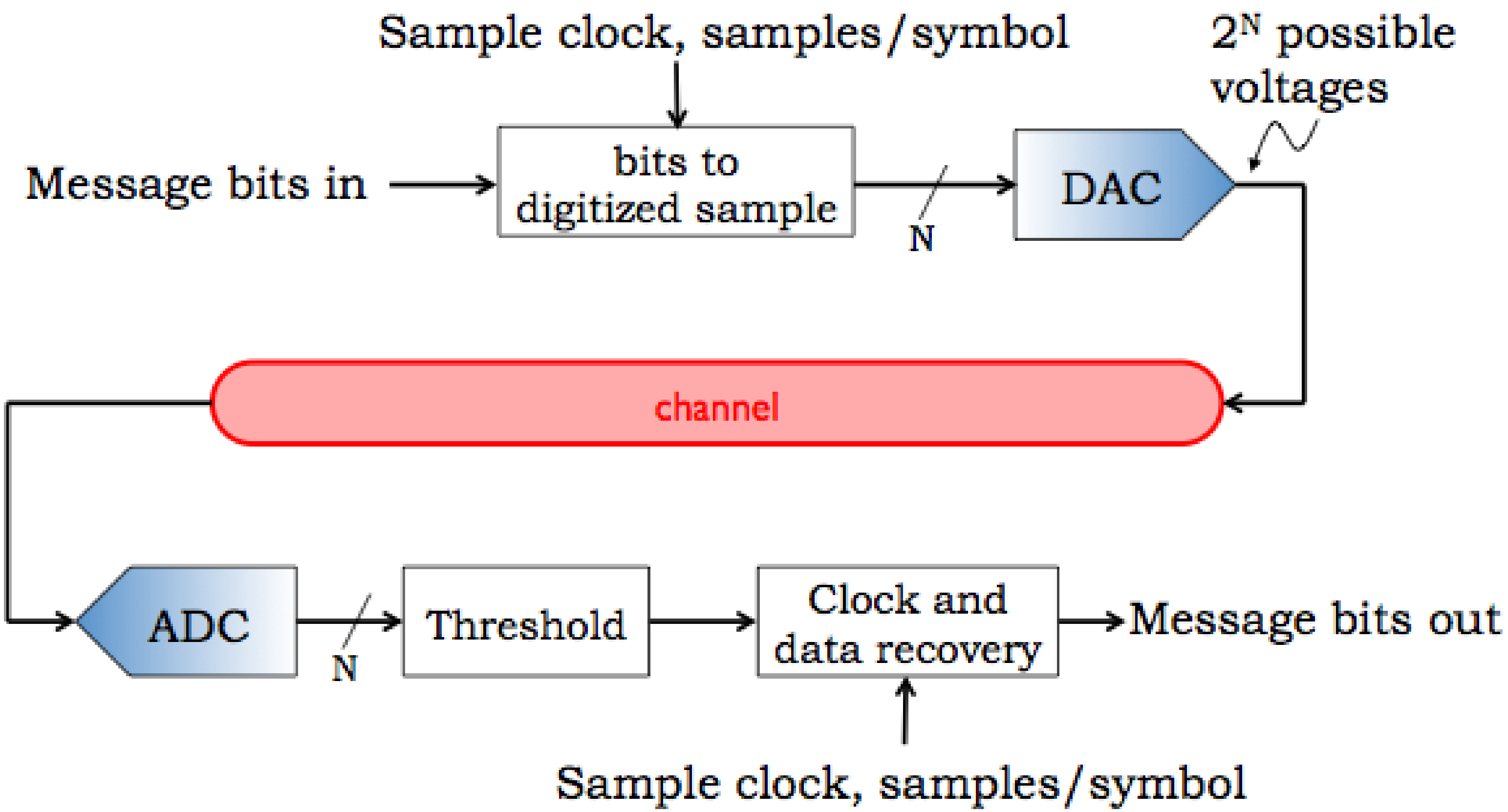
- Systems are classified according to such properties as: memory, causality, stability, linearity, and time-invariance.

- Linear, time-invariant systems are special systems for which a rich and powerful description is available. We will focus on such systems.

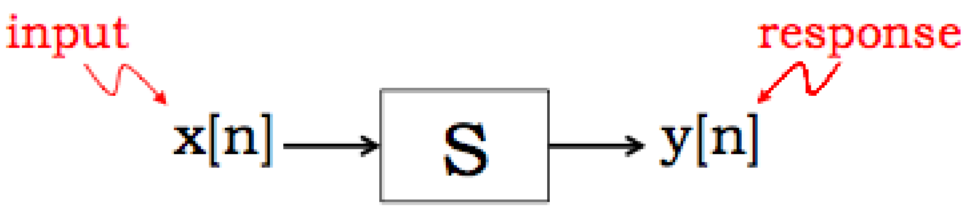
Week 5

Slide 84-105

Today: Modeling Channel Behavior



System Input and Response



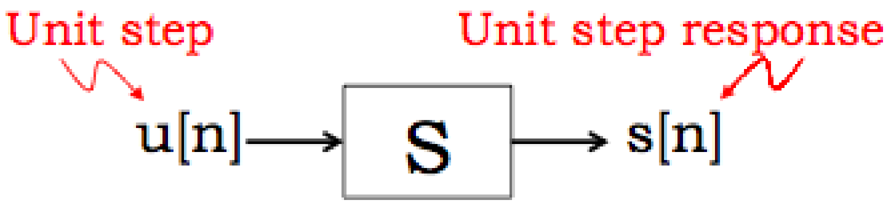
A discrete-time signal is described by an infinite sequence of values, denoted by x[n], y[n], z[n], and so on. The indices fall in the range $-\infty$ to $+\infty$.

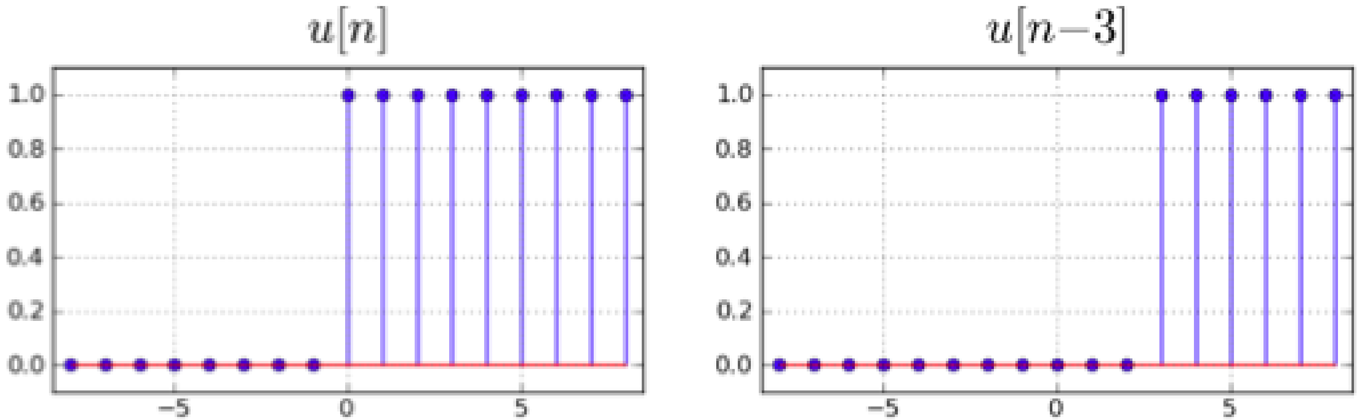
In the diagram above, the sequence of output values y[n] is called the *response* of system S to the *input* sequence x[n].

Unit Step and Unit Step Response

A simple but useful discrete-time signal is the *unit step*, u[n], defined as

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

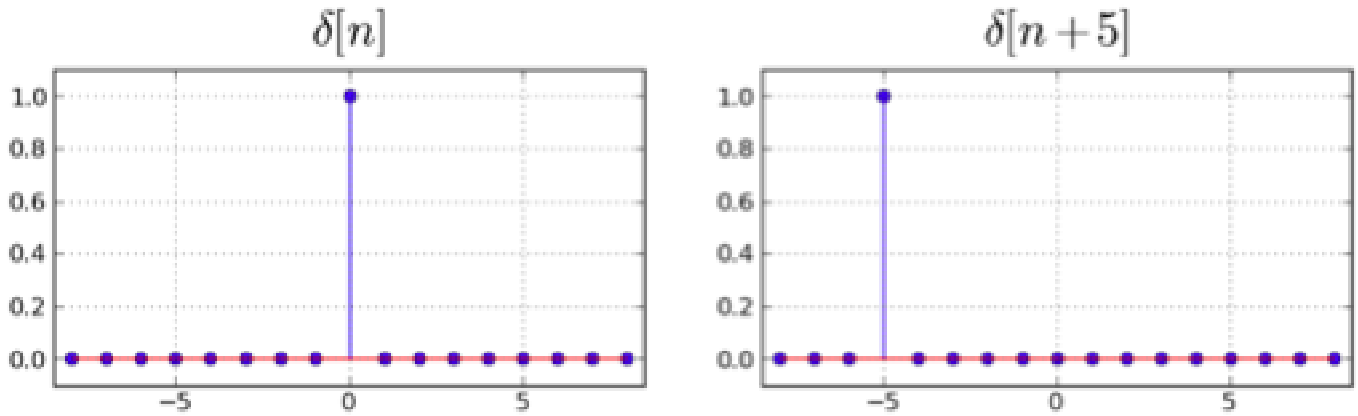




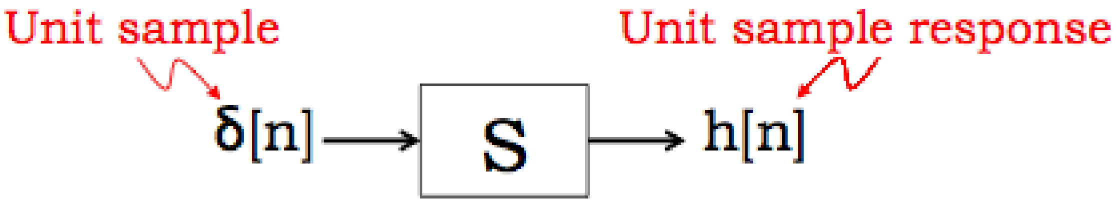
Unit Sample

Another simple but useful discrete-time signal is the *unit sample*, δ[n], defined as

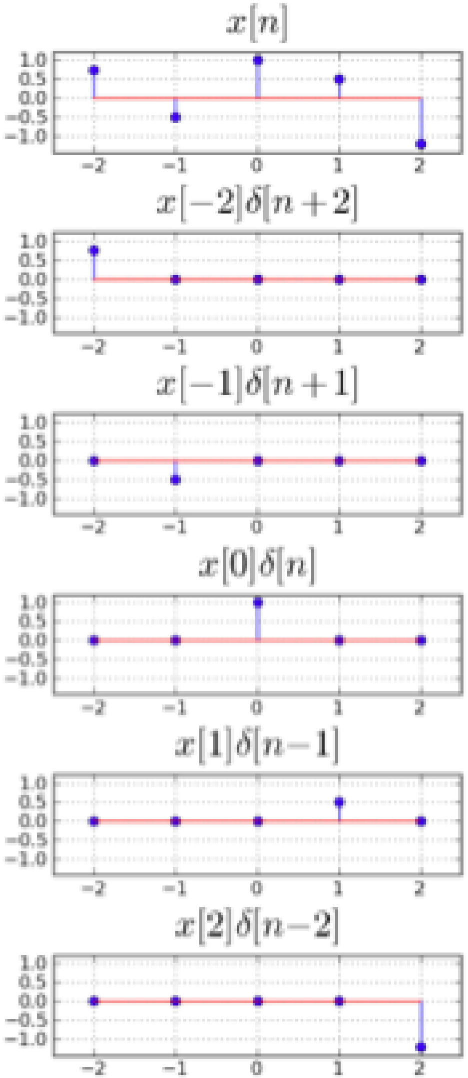
$$\delta[n] = u[n] - u[n-1] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



Unit Sample Response



The *unit sample response* of a system S is the response of the system to the unit sample input. We will always denote the unit sample response as h[n].



Unit-sample Decomposition

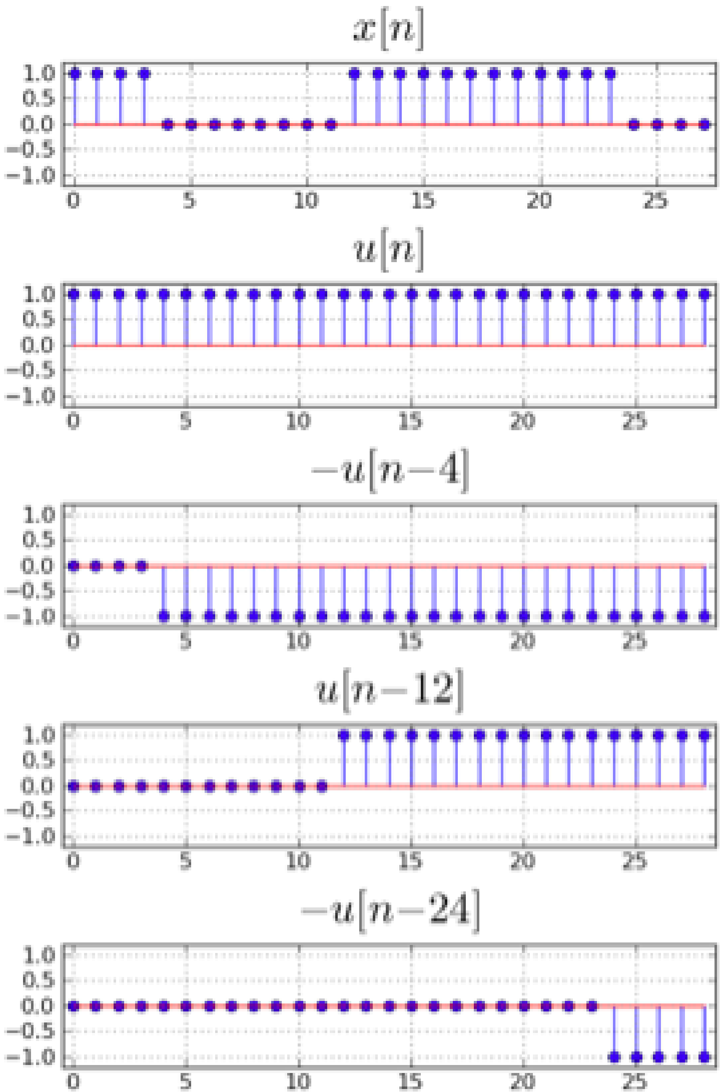
A discrete-time signal can be decomposed into a sum of time-shifted, scaled unit samples.

Example: in the figure, x[n] is the sum of x[-2]δ[n+2] + x[-1]δ[n+1] + … + x[2]δ[n-2].

In general:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

For any particular index, only one term of this sum is non-zero



Unit-step Decomposition

Digital signaling waveforms are easily decomposed into time-shifted, scaled unit steps (each transition corresponds to another shifted, scaled unit step).

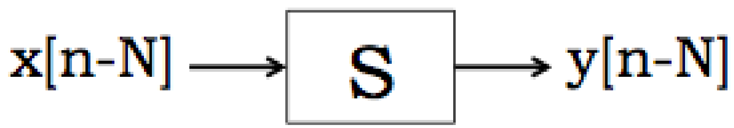
In this example, x[n] is the transmission of 1001110 using 4 samples/bit:

$$x[n] = u[n] - u[n-4] + u[n-12] - u[n-24]$$

Time Invariant Systems

Let y[n] be the response of S to input x[n].

If for all possible sequences x[n] and integers N



then system S is said to be *time invariant*. A time shift in the input sequence to S results in an identical time shift of the output sequence.

Linear Systems

Let $y_1[n]$ be the response of S to input $x_1[n]$ and $y_2[n]$ be the response to $x_2[n]$.

If

$$ax_1[n]+bx_2[n] \longrightarrow \boxed{S} \longrightarrow ay_1[n]+by_2[n]$$

then system S is said to be *linear*. If the input is the weighted sum of several signals, the response is the *superposition* (i.e., weighted sum) of the response to those signals.

Modeling LTI Systems

If system S is both linear and time-invariant (LTI), then we can use the unit sample response to predict the response to any input waveform x[n]:

Sum of shifted, scaled unit samples — Sum of shifted, scaled responses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{S} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Indeed, the unit sample response h[n] completely characterizes the LTI system S, so you often see

$$x[n] \longrightarrow \boxed{h_S[n]} \longrightarrow y[n]$$

Properties of Convolution

$$x[n] * h[n] \equiv \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The summation is called the convolution sum, or more simply, the *convolution* of x[n] and h[n]. "*" is the convolution operator.

Convolution is commutative:

$$x[n] * h[n] = h[n] * x[n]$$

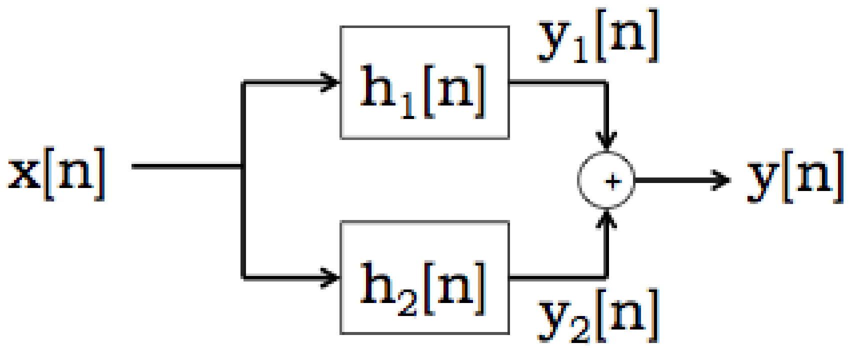
Convolution is associative:

$$x[n] * \left(h_1[n] * h_2[n]\right) = \left(x[n] * h_1[n]\right) * h_2[n]$$

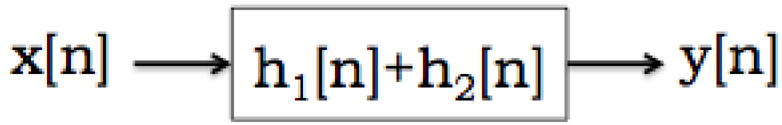
Convolution is distributive:

$$x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$$

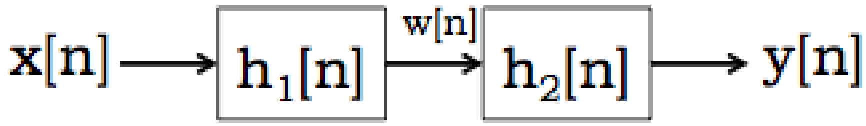
Parallel Interconnection of LTI Systems



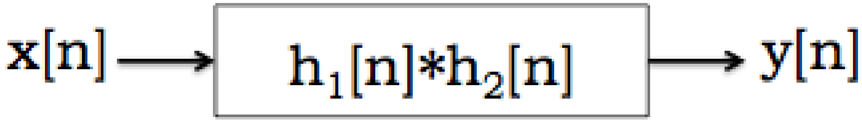
$$y[n] = y_1[n] + y_2[n] = x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \left(h_1[n] + h_2[n]\right)$$

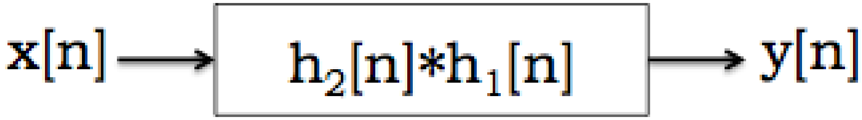


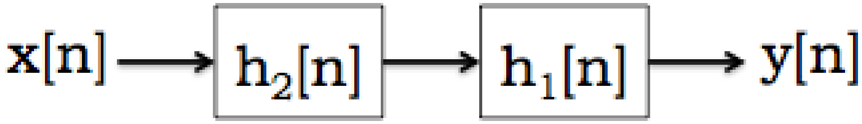
Series Interconnection of LTI Systems



$$y[n] = w[n] * h_2[n] = \left(x[n] * h_1[n]\right) * h_2[n] = x[n] * \left(h_1[n] * h_2[n]\right)$$







Channels as LTI Systems

Many transmission channels can be effectively modeled as LTI systems. When modeling transmissions, there are few simplifications we can make:

- We'll call the time transmissions start t=0; the signal before the start is 0. So x[m] = 0 for m < 0.
- Real-word channels are *causal*: the output at any time depends on values of the input at only the present and past times. So h[m] = 0 for m < 0.

These two observations allow us to rework the convolution sum when it's used to describe transmission channels:

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=\sum_{k=0}^{\infty} x[k]h[n-k]=\sum_{k=0}^{n} x[k]h[n-k]=\sum_{j=0}^{n} x[n-j]h[j]$$

start at t=0 causal j=n-k

Relationship between h[n] and s[n]

We're often given one of h[n] or s[n] and would like to know the other. On slide #5 we saw

$$\delta[n] = u[n] - u[n-1]$$

Which for LTI systems implies

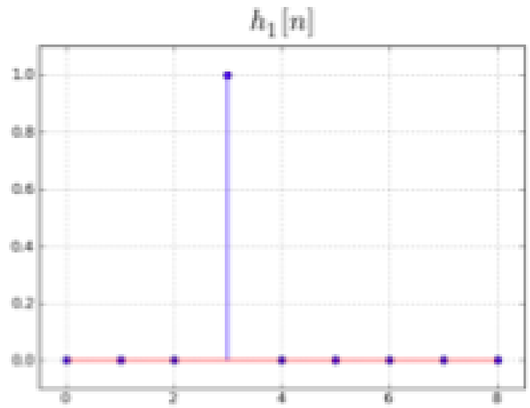
$$h[n] = s[n] - s[n-1]$$

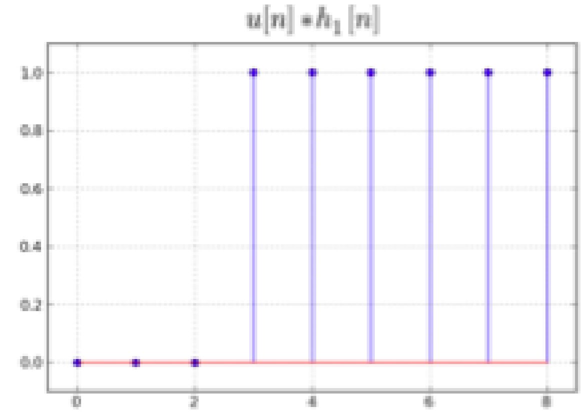
In other words, the unit sample response is the first difference of the unit step response. Also

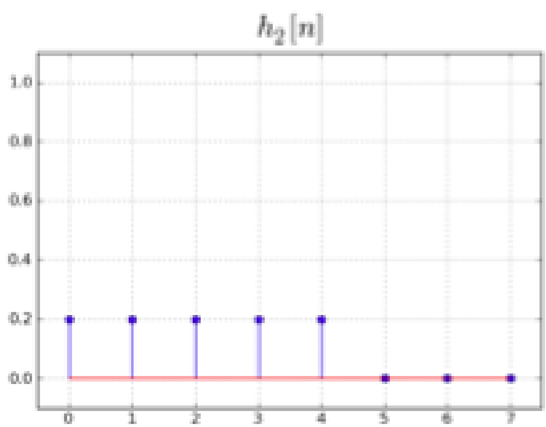
$$s[n] = \sum^{n} h[k]$$

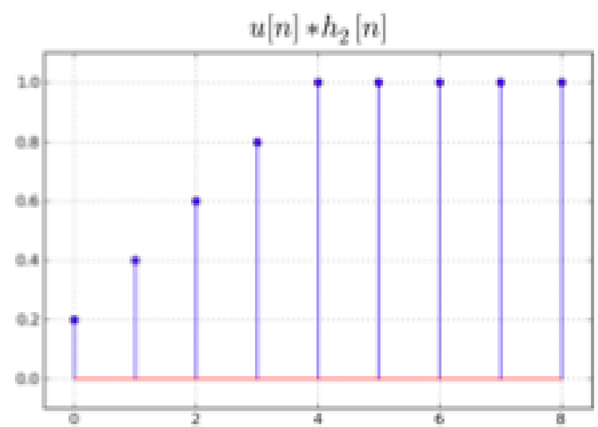
h[n]

s[n]=u[n]*h[n]

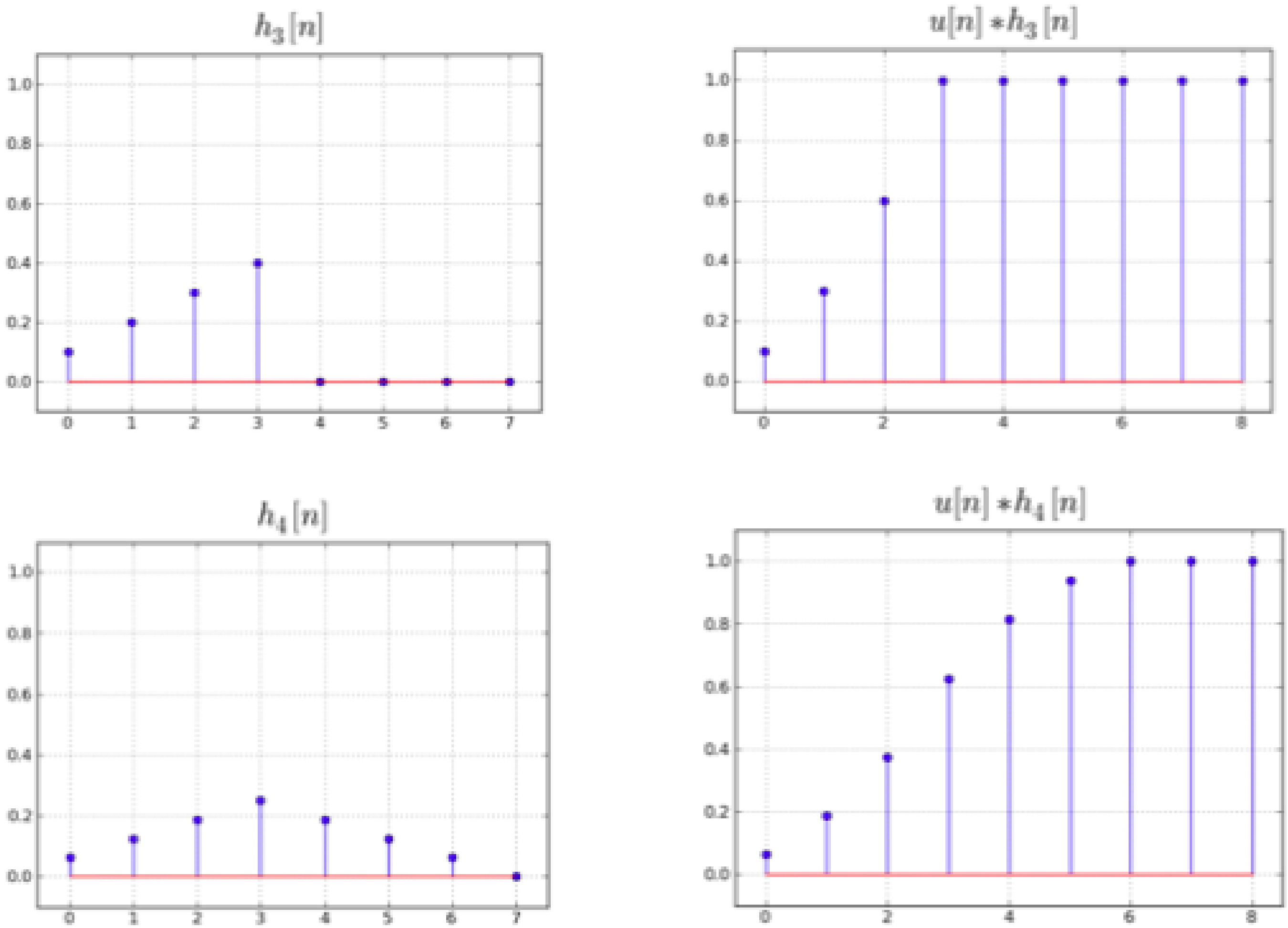




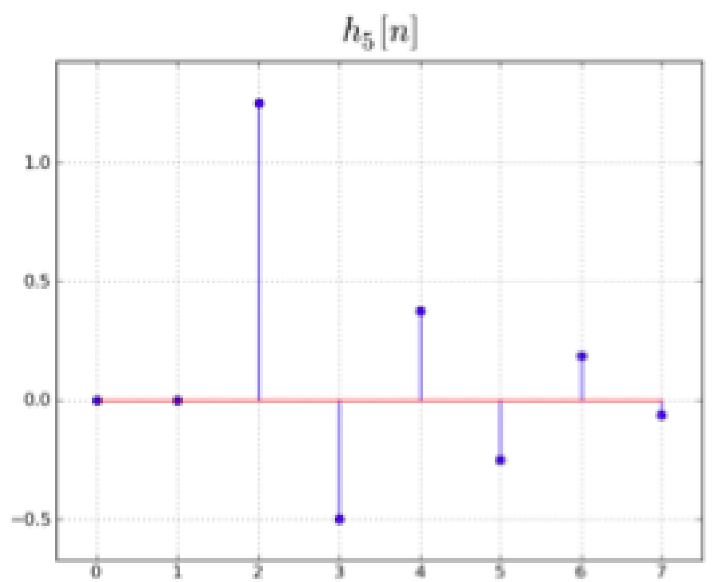


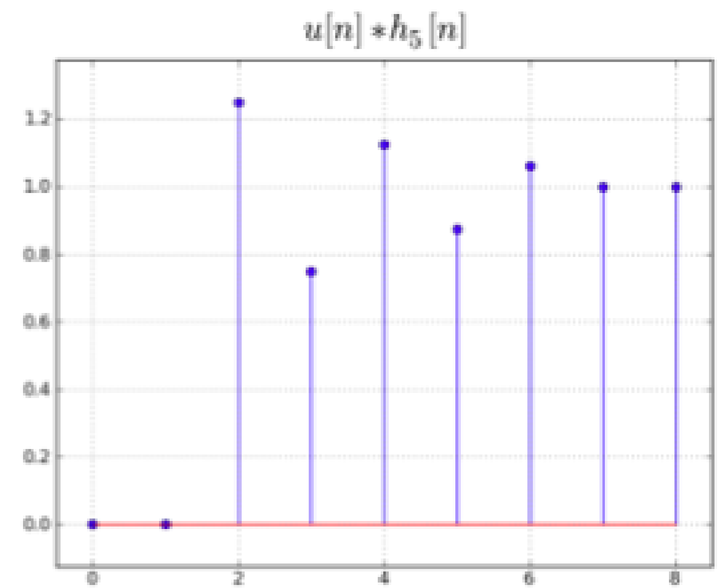


h[n]　　　　s[n]=u[n]*h[n]

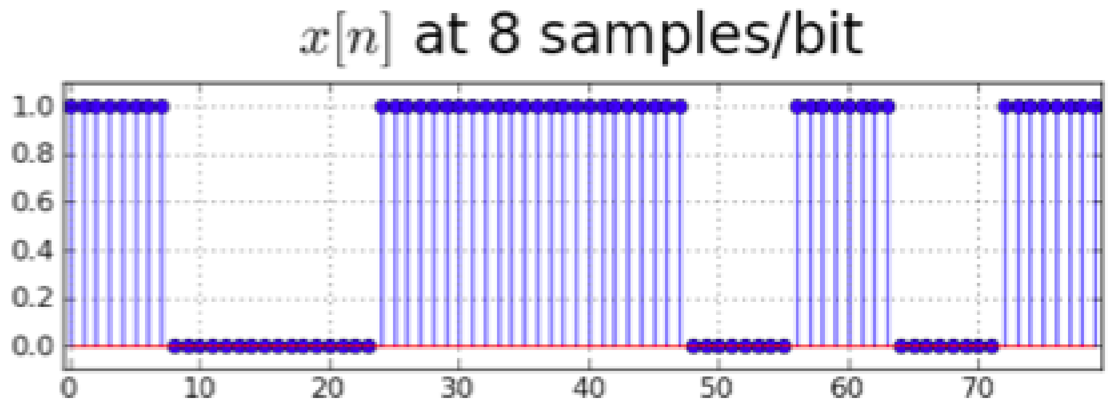


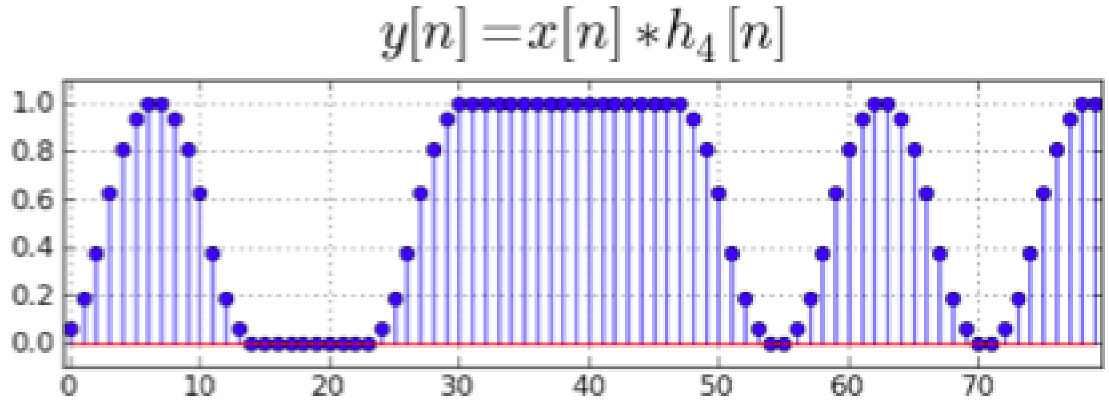
h[n]   s[n]=u[n]*h[n]





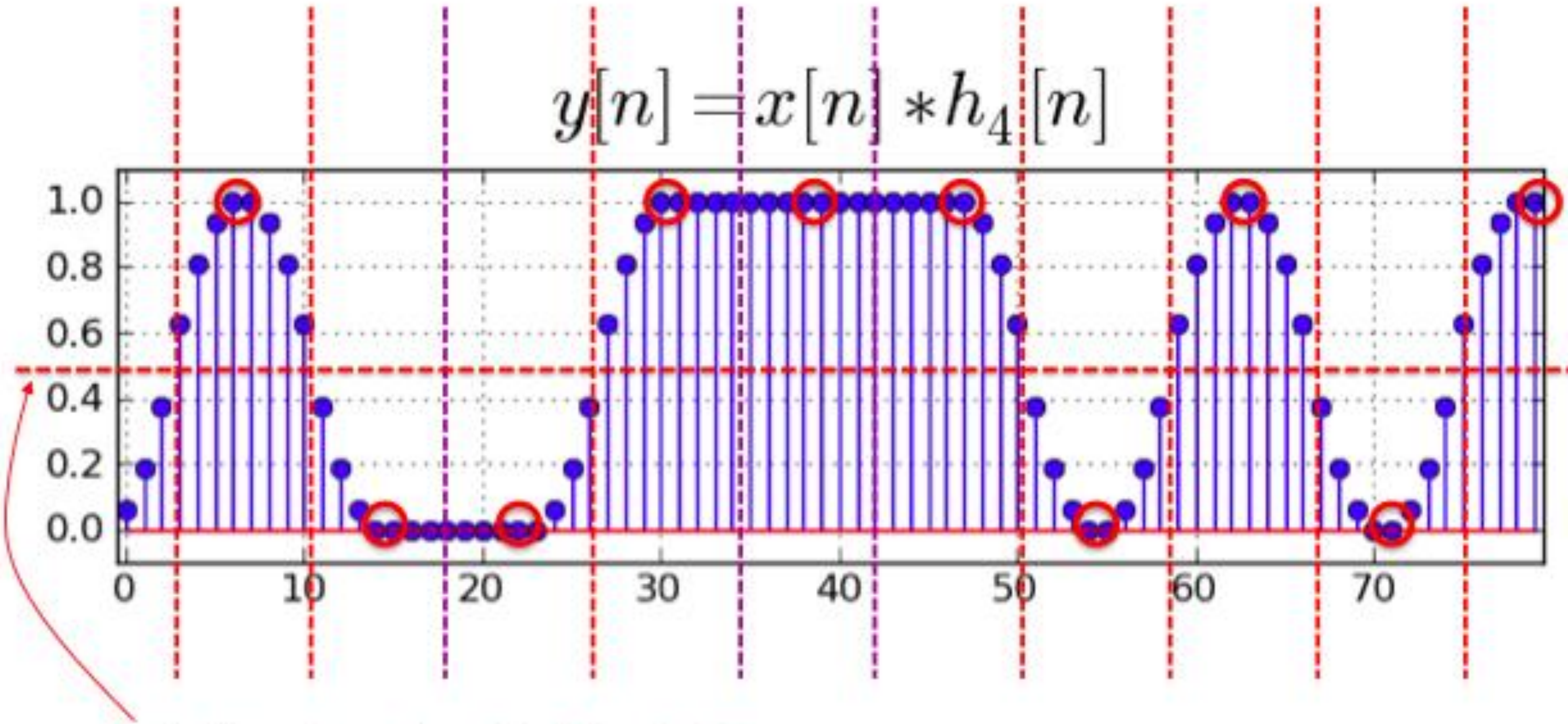
Transmission Over a Channel





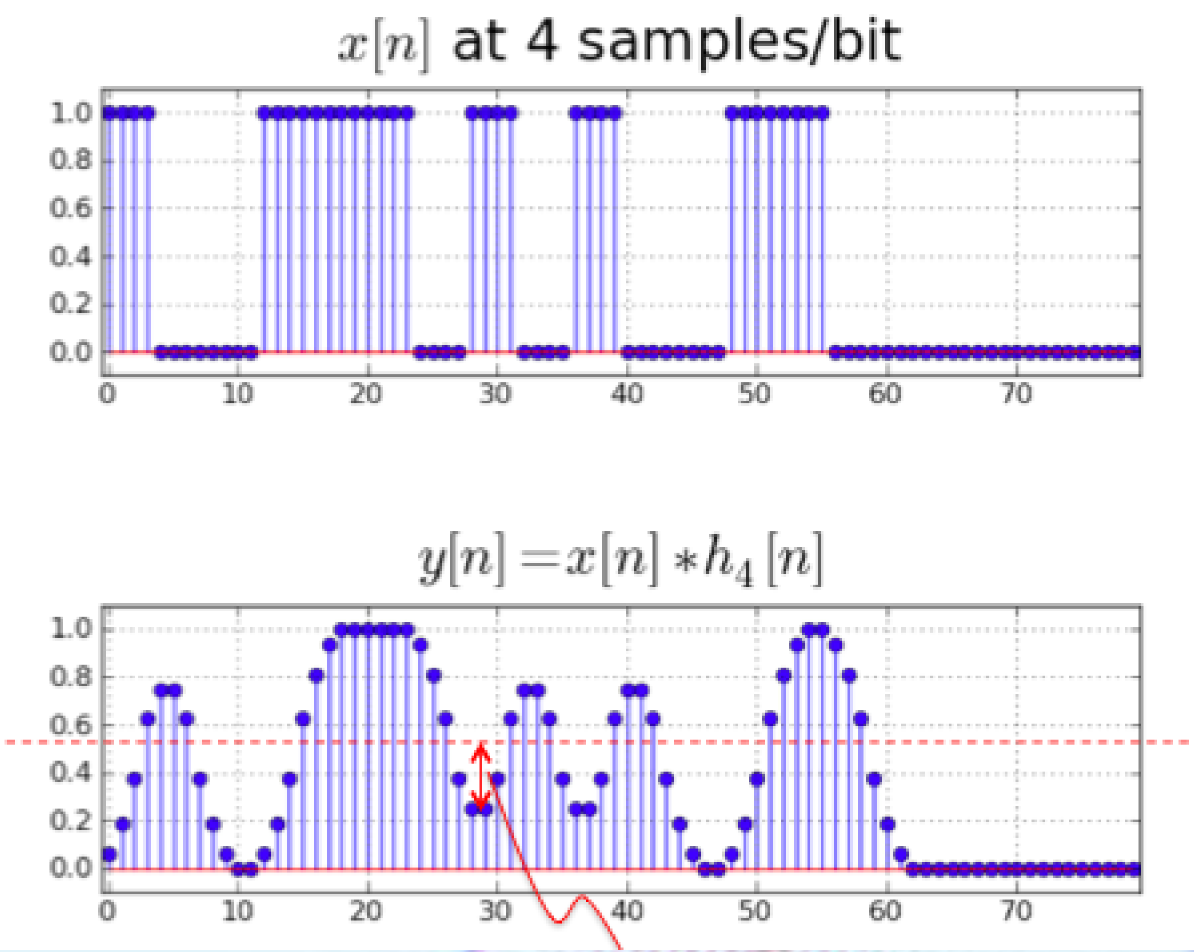
Receiving the Response

$$y[n] = x[n] * h_4[n]$$



Digitization threshold = 0.5V

Faster Transmission



Week 6

Slide 107-124

Amazing Property of LTI Systems

We will show that for a DT system

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m],$$

where $x[n]$ is an arbitrary input, $h[n]$ is the unit sample response, $y[n]$ is the output, and the above relation is called the superposition sum.

We will show that for a CT system

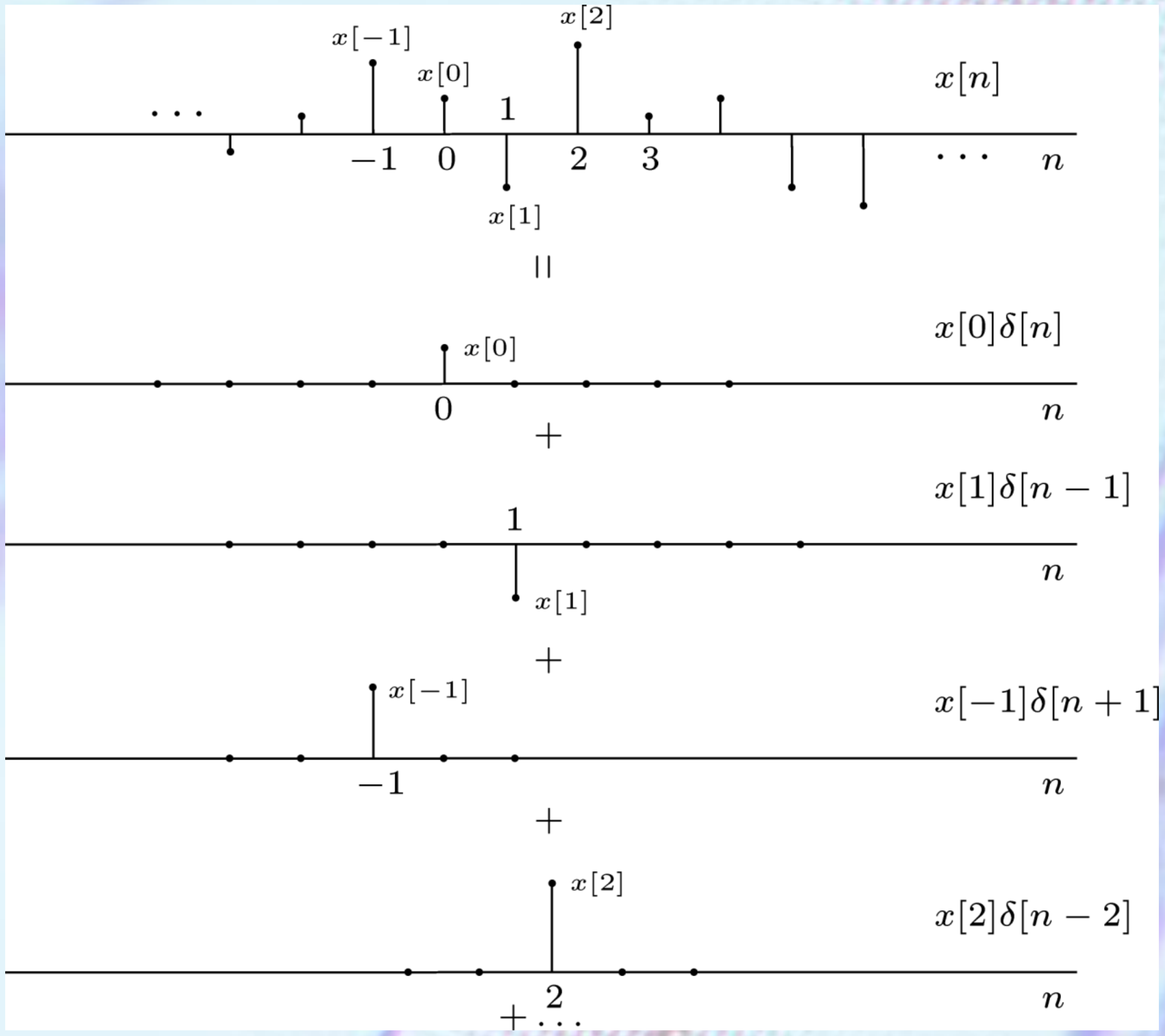
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau,$$

where $x(t)$ is an arbitrary input, $h(t)$ is the unit impulse response, $y(t)$ is the output, and the above relation is called the superposition integral.

Outline

- Superposition Sum for DT Systems
  - Representing Inputs as sums of unit samples
  - Using the Unit Sample Response
- Superposition Integral for CT System
  - Use limit of tall narrow pulse
- Unit Sample/Impulse Response and Systems
  - Causality, Memory, Stability

Representing DT Signals with Sums of Unit Samples



Written Analytically

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$$\Downarrow$$

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

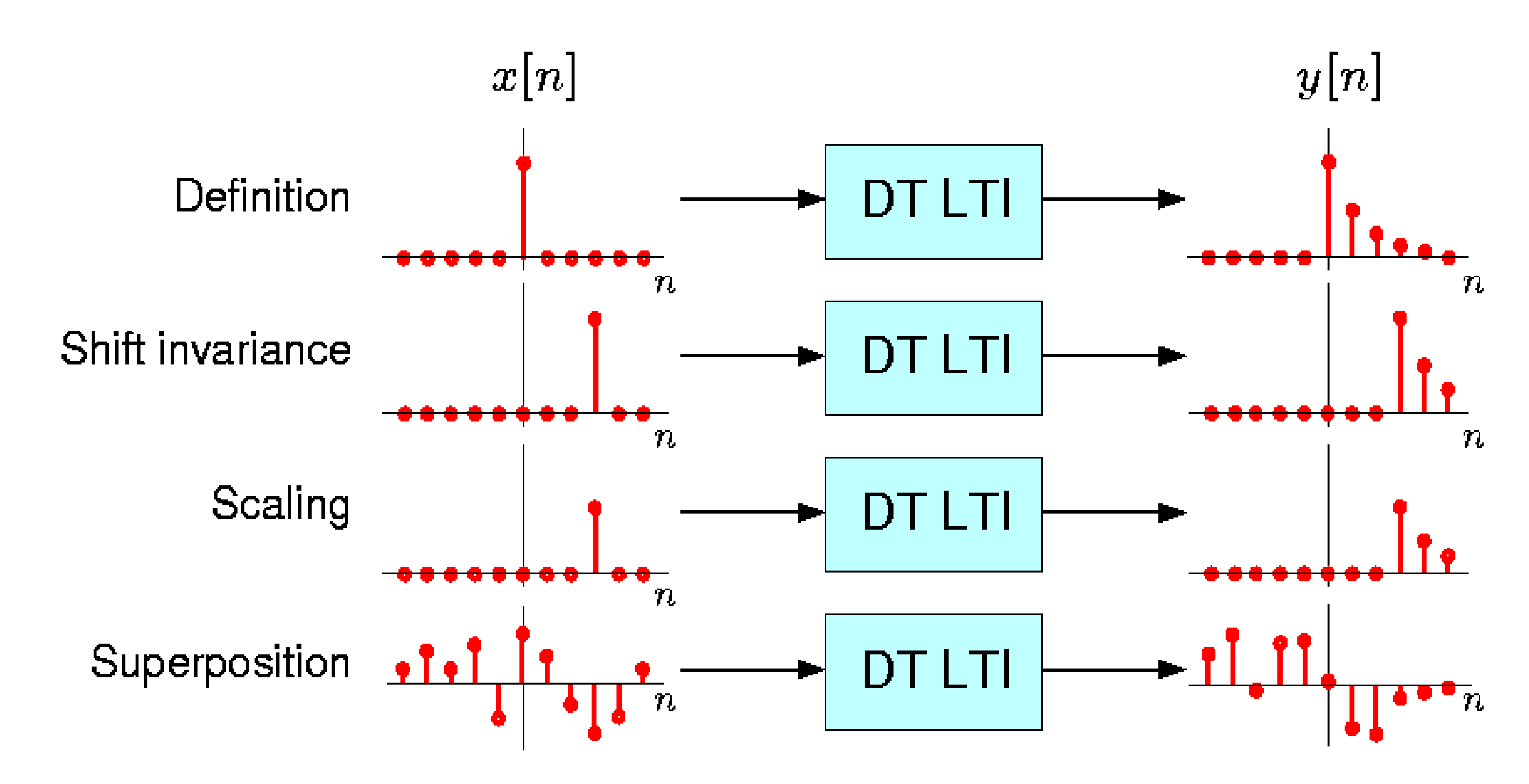
Coefficients

Basic Signals

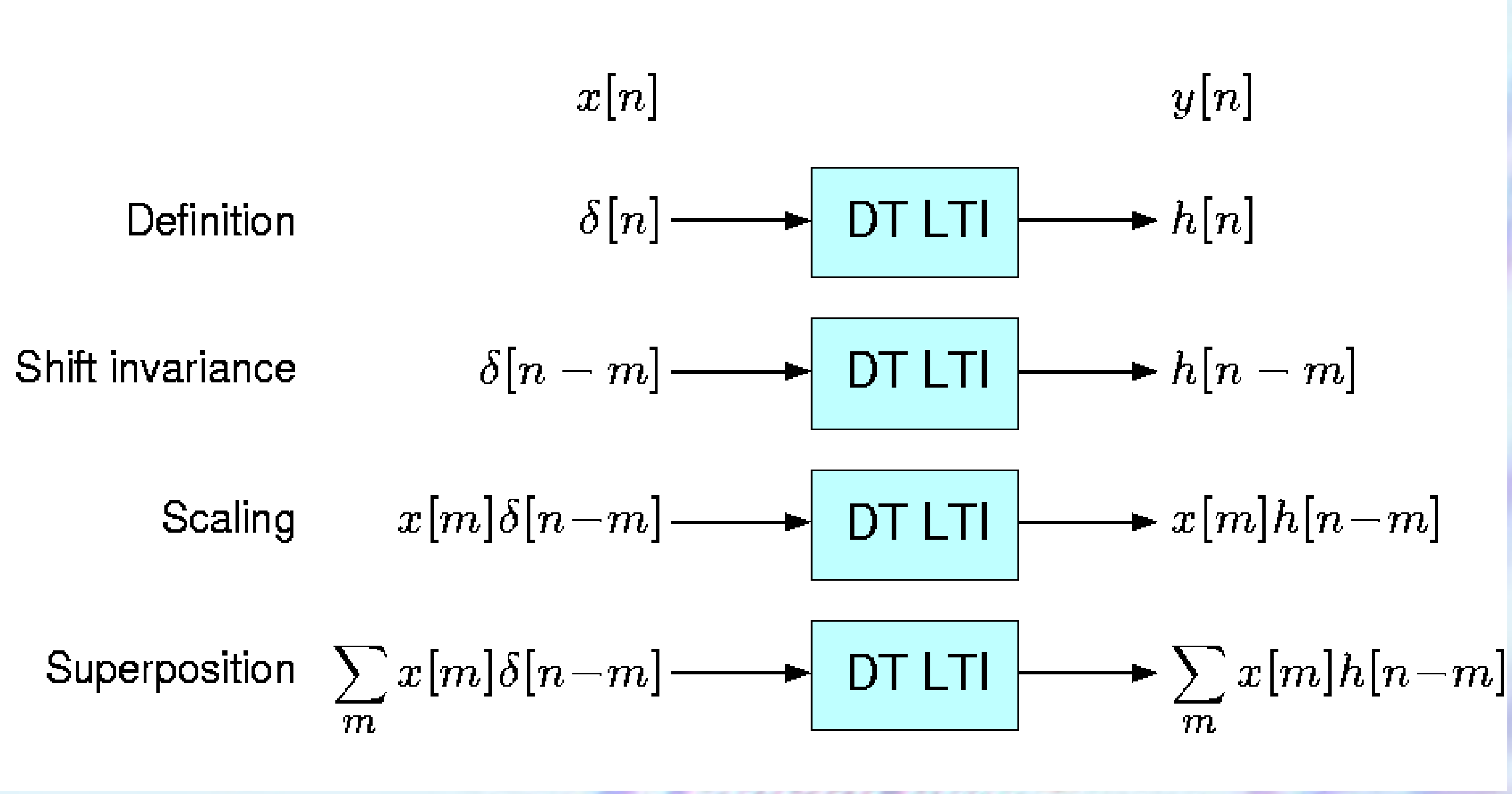
Note the Sifting Property of the Unit Sample

The Superposition Sum for DT Systems

Graphic View of Superposition Sum



Derivation of Superposition Sum



Convolution Sum

The relation

$$x[n] = \sum_m x[m]\delta[n-m]$$

expresses the sifting property of the unit sample. Note that the only non-zero term in this sum occurs when $m = n$, hence demonstrating the validity of the equation. The major conclusion of the derivation is that for an arbitrary input $x[n]$, the output is

$$y[n] = \sum_m x[m]h[n-m]$$

which is called the *superposition sum*. Such a relation is called a *convolution sum* when it involves arbitrary functions, i.e.,

$$z[n] = \sum_m x_1[m]x_2[n-m].$$

Thus, the superposition sum is a special case of the convolution sum.

Convolution Notation

We shall write the convolution sum of two DT signals as

$$z[n] = x_1[n] * x_2[n] = \sum_m x_1[m]x_2[n-m].$$

The symbol for convolution in various textbooks includes

$$x_1[n] * x_2[n],\ x_1[n] \star x_2[n], \text{ and } x_1[n] \otimes x_2[n].$$

Notation is confusing, should not have [n]

takes two *sequences* and produces a third *sequence*

$z = x_1 * x_2$ makes more sense

Learn to live with it.

Convolution Computation Mechanics

$$y[n] = x[n] * h[n] = \sum_m x[m]h[n-m],$$

Step 1 Plot x and h vs m since the convolution sum is on m.

Step 2 Flip $h[m]$ around the vertical axis to obtain $h[-m]$.

Step 3 Shift $h[-m]$ by n to obtain $h[n-m]$.

Step 4 Multiply to obtain $x[m]h[n-m]$.

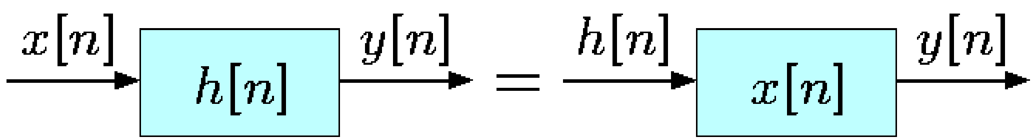
Step 5 Sum on m to compute $\sum_m x[m]h[n-m]$.

Step 6 Index n and repeat Steps 3-6.

DT Convolution Properties

Commutative Property

$$x[n] * h[n] = h[n] * x[n]$$



Proof:

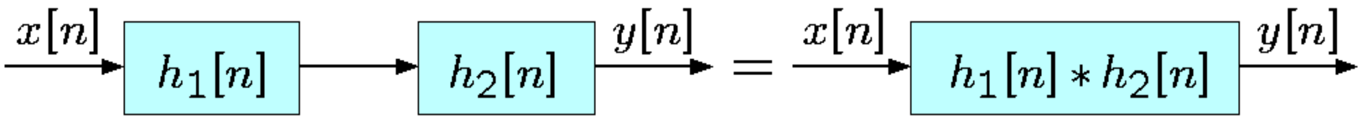
$$y[n] = x[n] * h[n] = \sum_m x[m]h[n-m].$$

Let $n - m = l$ then

$$y[n] = x[n] * h[n] = \sum_l x[n-l]h[l] = \sum_l h[l]x[n-l] = h[n] * x[n].$$

Associative Property

$$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$



Proof:

$$y[n] = \sum_l \left(\sum_m x[m] h_1[l-m] \right) h_2[n-l] = \sum_m x[m] \left(\sum_l h_1[l-m] h_2[n-l] \right).$$

Let $k = l - m$ to obtain

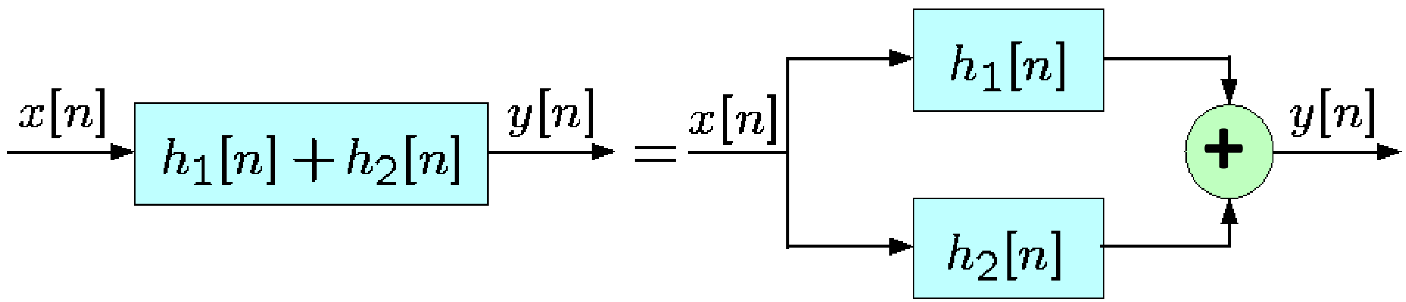
$$y[n] = \sum_m x[m] \left(\sum_k h_1[k] h_2[n-m-k] \right) = \sum_m x[m] h[n-m],$$

where

$$h[n] = h_1[n] * h_2[n].$$

Distributive Property

$$y[n] = x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$



Proof:

$$\begin{aligned} y[n] &= \sum_m x[m](h_1[n-m] + h_2[n-m]) \\ &= \sum_m x[m]h_1[n-m] + \sum_m x[m]h_2[n-m] \end{aligned}$$

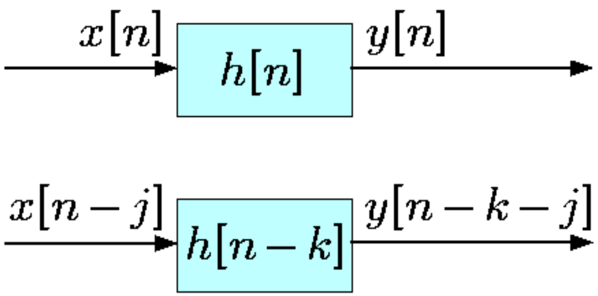
Delay Accumulation

If

$$y[n] = x[n] * h[n]$$

then

$$x[n-j] * h[n-k] = y[n-k-j]$$

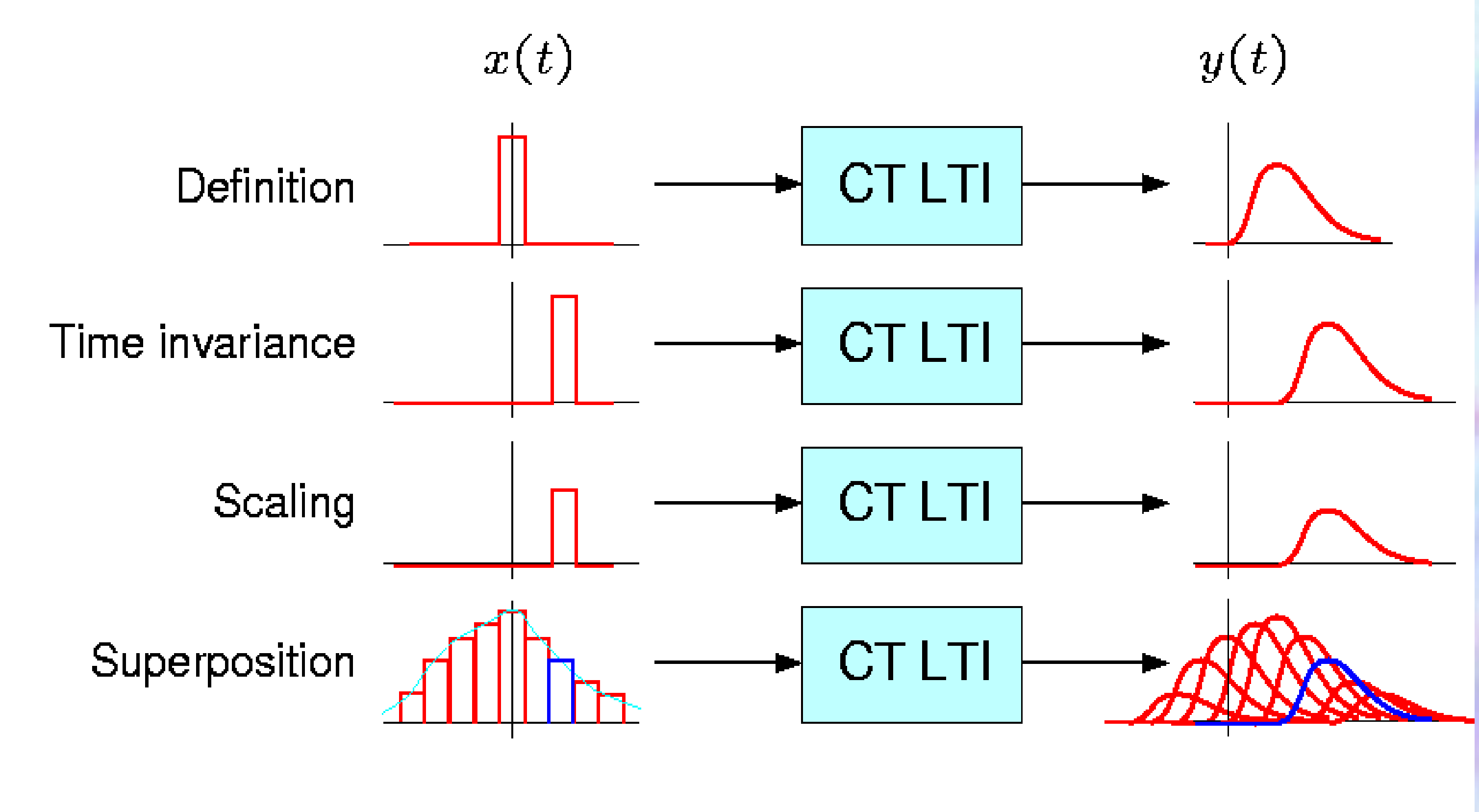


Proof:

$$\sum_m x[m-j]h[n-k-m] = \sum_l x[l]h[n-k-j-l] = y[n-k-j].$$

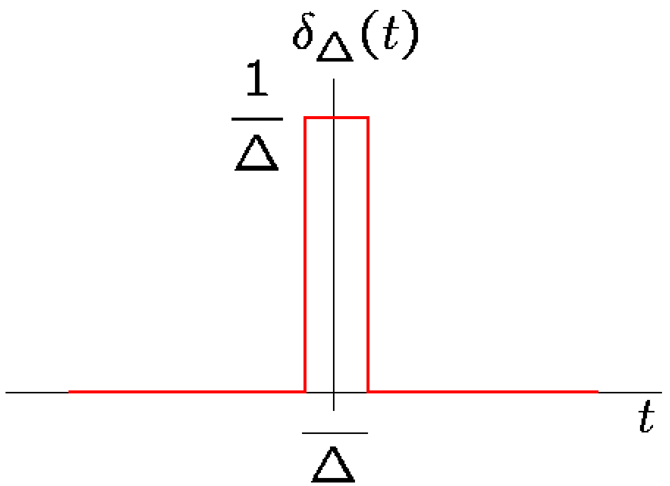
Superposition Integral for CT Systems

Graphic View of Staircase Approximation



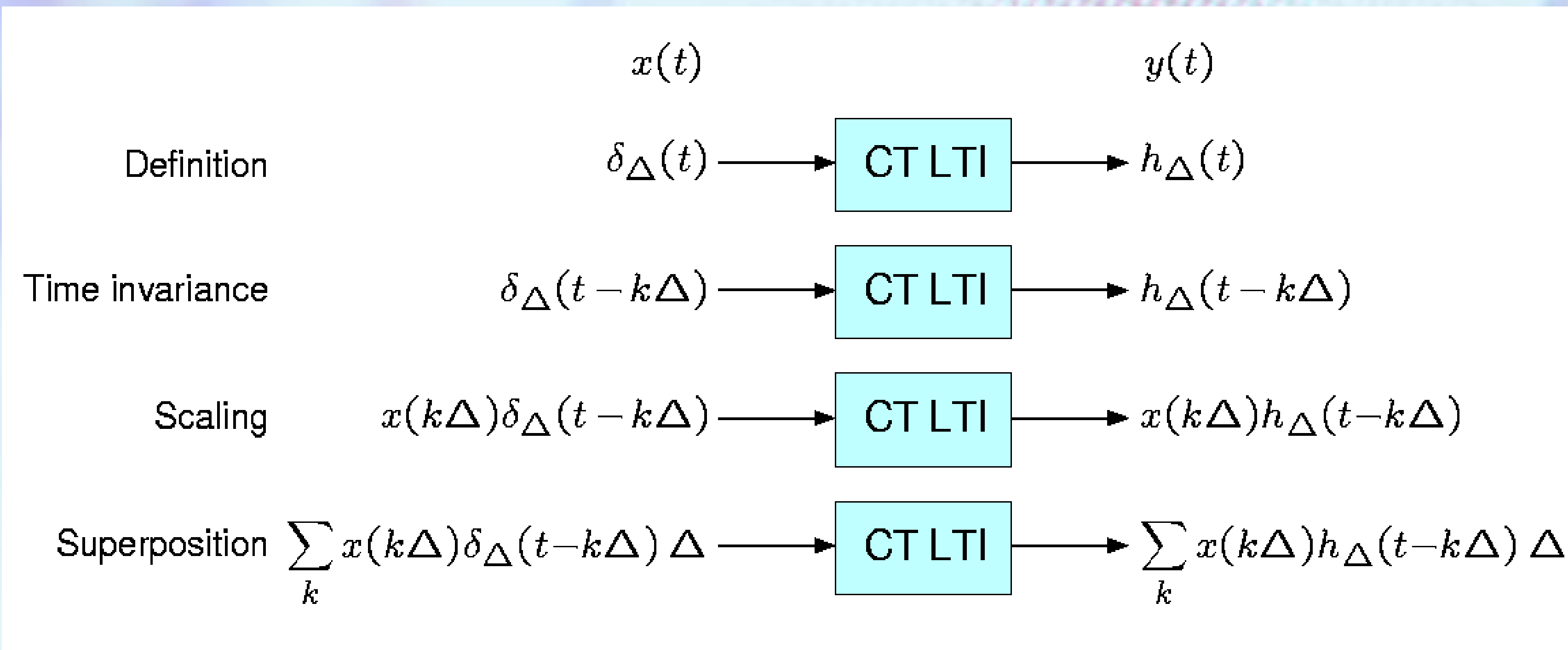
Tall Narrow Pulse

To represent a continuous time function with a staircase approximation, it is useful to define a tall and narrow pulse.



As Δ is decreased, the width of the pulse decreases and the amplitude increases while the area remains 1.

Derivation of Staircase Approximation of Superposition Integral



The Superposition Integral

The derivation shows that a staircase approximation to the input $x(t)$

$$x_\Delta(t) = \sum_k x(k\Delta)\delta_\Delta(t - k\Delta)\,\Delta$$

yields a staircase approximation to the output

$$y_\Delta(t) = \sum_k x(k\Delta)h_\Delta(t - k\Delta)\,\Delta.$$

Now we take the limit as $\Delta \to 0$, $k \to \infty$, $k\Delta = \tau$, $x_\Delta(t) \to x(t)$, $h_\Delta(t-k\Delta) \to h(t-\tau)$, and $\delta_\Delta(t-k\Delta) \to \delta(t-\tau)$ in a generalized function sense. Then the sums approach the integrals

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\,d\tau,$$

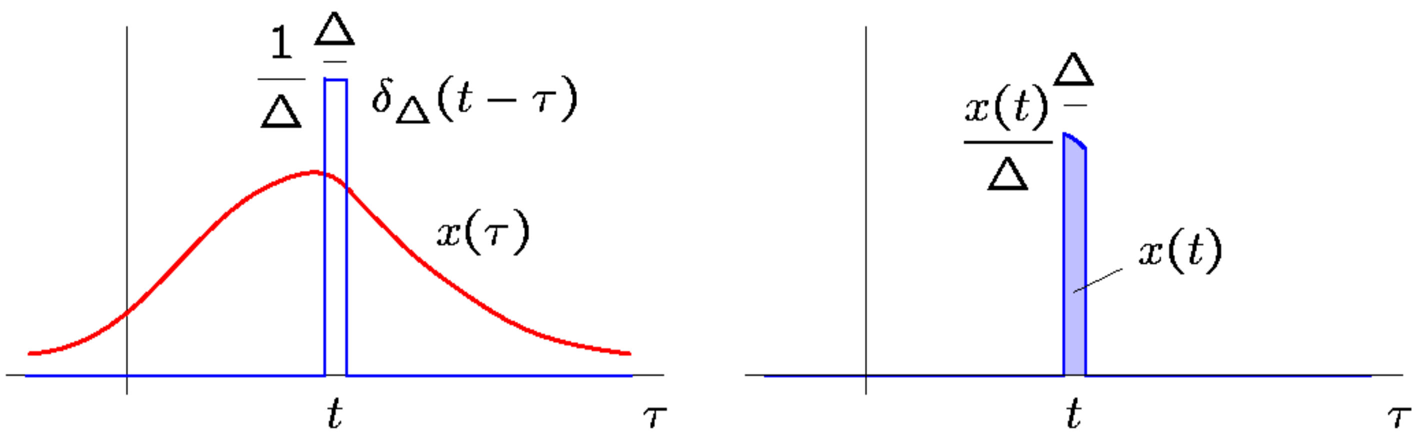
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\,d\tau.$$

Sifting Property of Unit Impulse

We examine the sifting property of the unit impulse which is inherent in the expression

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau.$$

To see what this means we approximate the impulse with a tall, narrow pulse.



Week 7

Slide 126-138

CT Convolution Mechanics

To compute the superposition integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau,$$

Step 1 Plot x and h vs τ since the convolution integral is on τ.

Step 2 Flip $h(\tau)$ around the vertical axis to obtain $h(-\tau)$.

Step 3 Shift $h(\tau)$ by t to obtain $h(t-\tau)$.

Step 4 Multiply to obtain $x(\tau)h(t-\tau)$.

Step 5 Integrate on τ to compute $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$.

Step 6 Increase t and repeat Steps 3-6.

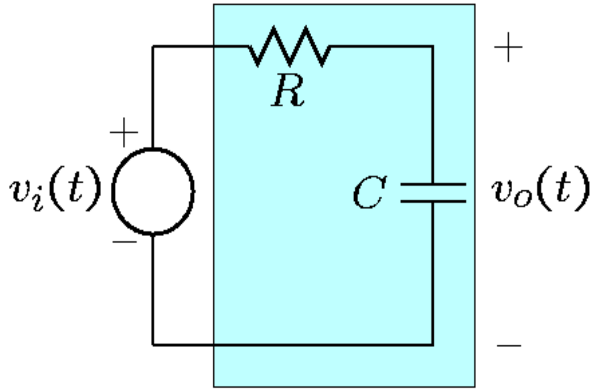
CT Convolution Properties

Commutative	$y(t) = x(t) * h(t) = h(t) * x(t)$
Associative	$y(t) = (x(t) * h_1(t)) * h_2(t)$
	$y(t) = x(t) * (h_1(t) * h_2(t))$
Distributive	$y(t) = x(t) * (h_1(t) + h_2(t))$
	$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$
Delay accumulation	$y(t - \tau_1 - \tau_2) = x(t - \tau_1) * h(t - \tau_2)$
Derivative accumulation	$y^{[n+m]}(t) = x^{[n]}(t) * h^{[m]}(t)$

The last property implies that differentiating the input n times and the impulse response m times results in an output that is differentiated $n + m$ times.

Computing Unit Sample/Impulse Responses Circuit Example

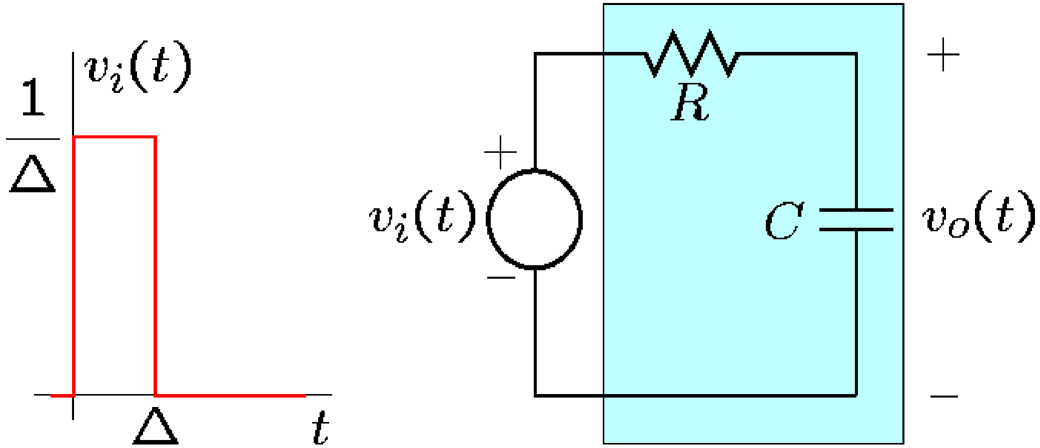
The unit sample/impulse response of an LTI system characterizes that system. How can we measure this response on a real system? To indicate how this might be done we will determine the impulse response of a CT LTI system that is a lowpass filter.



To find the impulse response we will determine the response to a tall, narrow pulse.

Narrow pulse approach

Determine $v_o(t)$ in response to the pulse of input shown.



Narrow pulse response

Note that $v_i(t)$ can be written as the difference of a step and a step delayed that is scaled as follows

$$v_i(t) = \frac{1}{\Delta}(u(t) - u(t - \Delta)).$$

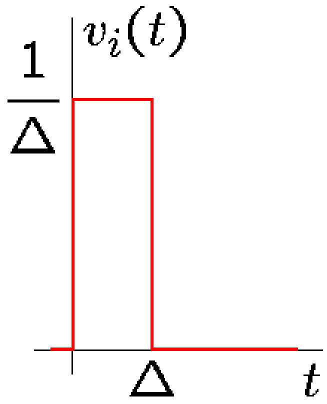
Therefore, we need only find the response of the network to a unit step $v_i(t) = u(t)$. We denote the step response as $v_o(t) = s(t)$ which is

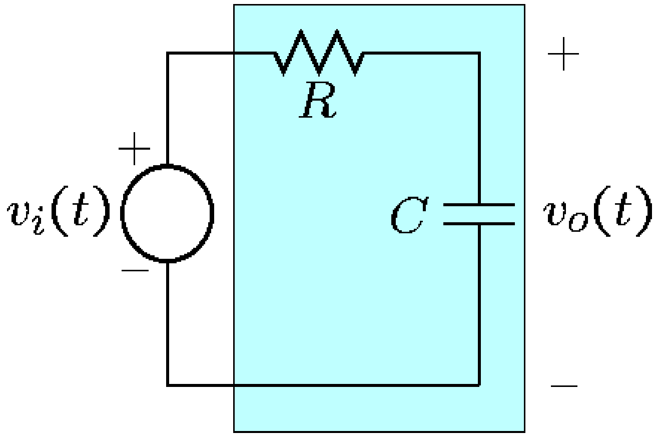
$$s(t) = (1 - e^{-\alpha t})u(t),$$

where $\alpha = 1/(RC)$. Therefore, the response to the pulse is

$$v_o(t) = \frac{1}{\Delta}\left((1 - e^{-\alpha t})u(t) - (1 - e^{-\alpha(t-\Delta)})u(t - \Delta)\right)$$

Narrow pulse response cont’d

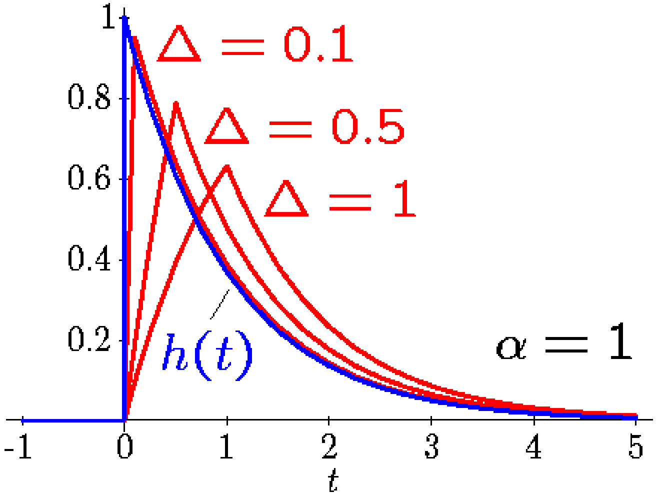




The pulse response can be written as

$$v_o(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{\Delta}(1 - e^{-\alpha t}) & \text{for } 0 \leq t < \Delta \\ \frac{1}{\Delta}(e^{\alpha\Delta} - 1)e^{-\alpha t} & \text{for } t \geq \Delta \end{cases}$$

Convergence of Narrow pulse response



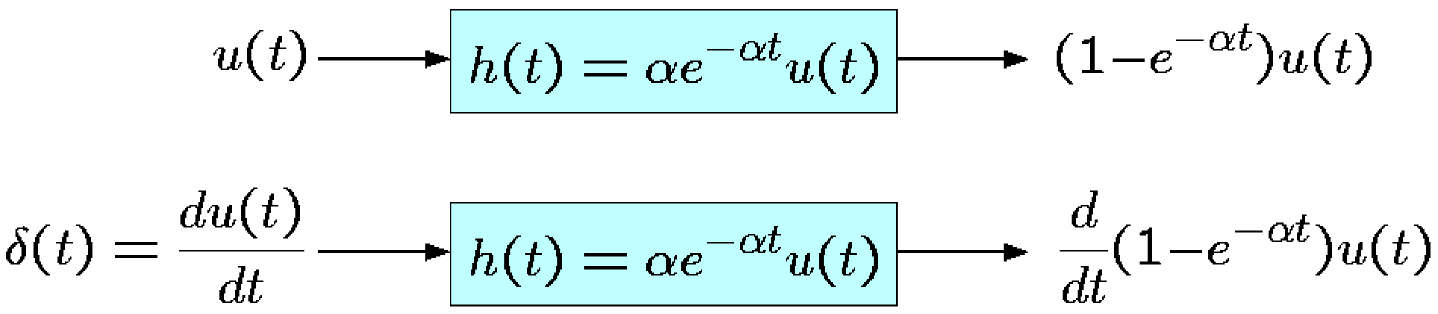
As $\alpha\Delta \to 0$

$$v_o(t) \to \begin{cases} 0 & \text{for } t < 0 \\ \frac{\alpha}{\Delta} t & \text{for } 0 \le t < \Delta \\ \alpha e^{-\alpha t} & \text{for } t \ge \Delta \end{cases}$$

Hence, the impulse response of the LPF is

$$h(t) = \alpha e^{-\alpha t} u(t).$$

Alternative Approach – Use Differentiation



To determine the impulse response, we need to evaluate the derivative which we do by parts

$$h(t) = \frac{d}{dt}\Big((1 - e^{-\alpha t})u(t)\Big) = \alpha e^{-\alpha t}u(t) + (1 - e^{-\alpha t})\delta(t).$$

Alternative Approach – Use Differentiation cont’d

To simplify the impulse response we need to interpret the term $(1 - e^{-\alpha t})\delta(t)$ which we do by placing that term in an integral and noting that

$$\int_{\infty}^{\infty} (1 - e^{-\alpha t})\delta(t)\, dt = (1 - e^{-\alpha t})\Big|_{t=0} = 0.$$

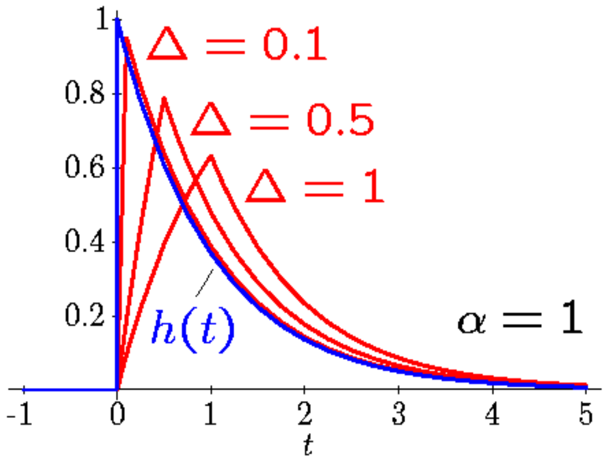
Therefore, the second term is an impulse of area 0 which equals 0, so that the impulse response is

$$h(t) = \alpha e^{-\alpha t} u(t).$$

This agrees with the result obtained by finding the response to a tall, narrow pulse.

How to measure Impulse Responses

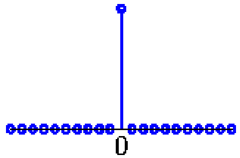
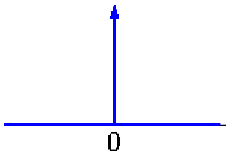
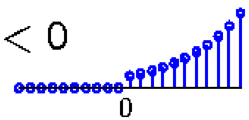
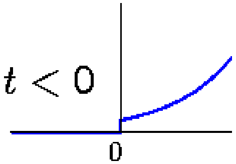
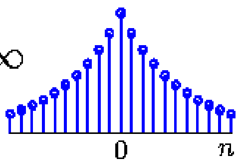
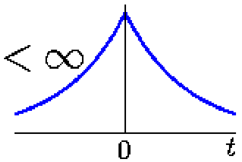
Apply a brief rectangular pulse and measure the response. Repeat the measurement with a pulse of briefer duration but the same area.



Response of the LPF to pulses of different durations.

Repeat the process with briefer pulses until the changes in the pulse responses do not matter to you. The pulse response to the briefest of these pulses is an estimate of the impulse response.

Unit Sample/Impulse Responses of Different Classes of Systems

Memoryless	$h[n] = \delta[n]$		$h(t) = \delta(t)$	
Causal	$h[n] = 0$ for $n < 0$		$h(t) = 0$ for $t < 0$	
BIBO stable	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$		$\int_{-\infty}^{\infty} \lvert h(t) \rvert \, dt < \infty$	

The condition for the unit impulse/sample response for a BIBO stable system requires some justification.

Bounded-Input Bounded-Output Stability

$y[n]$ can be expressed in terms of the input $x[n]$ as

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

For a BIBO stable system, if $x[n] < \infty$ then $y[n] < \infty$. Therefore,

$$\left| \sum_{m=-\infty}^{\infty} h[m]x[n-m] \right| < \sum_{m=-\infty}^{\infty} |h[m]||x[n-m]| < x_{max} \sum_{m=-\infty}^{\infty} |h[m]|,$$

where x_{max} is the maximum value of $|x[m]|$. Therefore, if $x_{max} < \infty$, and if

$$\sum_{m=-\infty}^{\infty} |h[m]| < \infty,$$

then $y[n] < \infty$. This condition is both necessary and sufficient.

Conclusions

- Time functions can be represented as superpositions of unit samples/impulses,

$$x[n] = \sum_m x[m]\delta[n-m] \text{ and } x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau.$$

- The unit sample response of an LTI DT system $h[n]$ and the unit impulse response of an LTI CT system $h(t)$ characterize those systems.

- The superposition sum and integral allow a computation of the output for an arbitrary input given only the unit sample/impulse response,

$$y[n] = \sum_m x[m]h[n-m] \text{ and } y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

Week 8

Slide 140-156

Fourier Series & The Fourier Transform

What is the Fourier Transform?

Anharmonic Waves

Fourier Cosine Series for even functions

Fourier Sine Series for odd functions

The continuous limit: the Fourier transform (and its inverse)

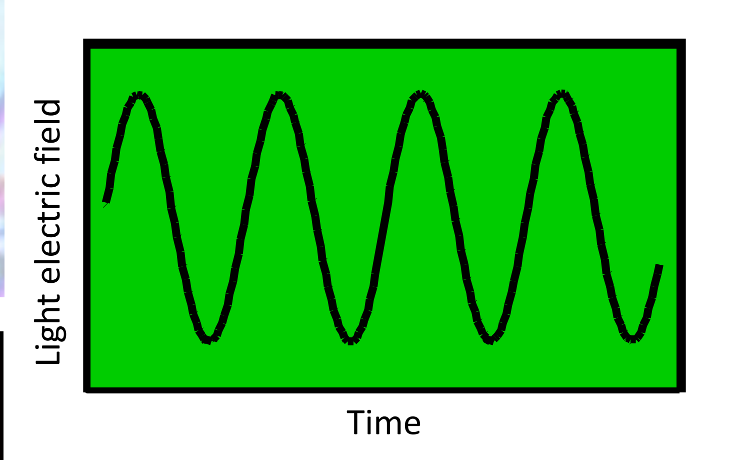
Some transform examples

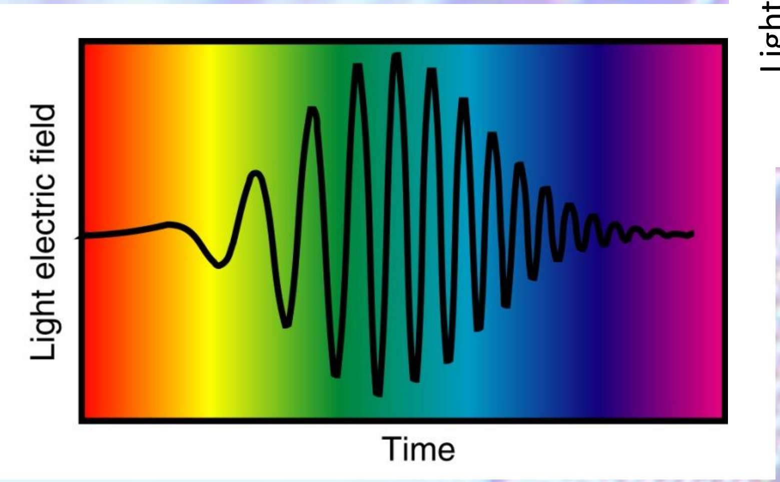
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

What do we hope to achieve with the Fourier Transform?

We desire a measure of the frequencies present in a wave. This will lead to a definition of the term, the **spectrum**.

Plane waves have only one frequency, ω.





This light wave has many frequencies. And the frequency increases in time (from red to blue).

It will be nice if our measure also tells us **when** each frequency occurs.

Lord Kelvin on Fourier' s theorem

Fourier' s theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.

Lord Kelvin

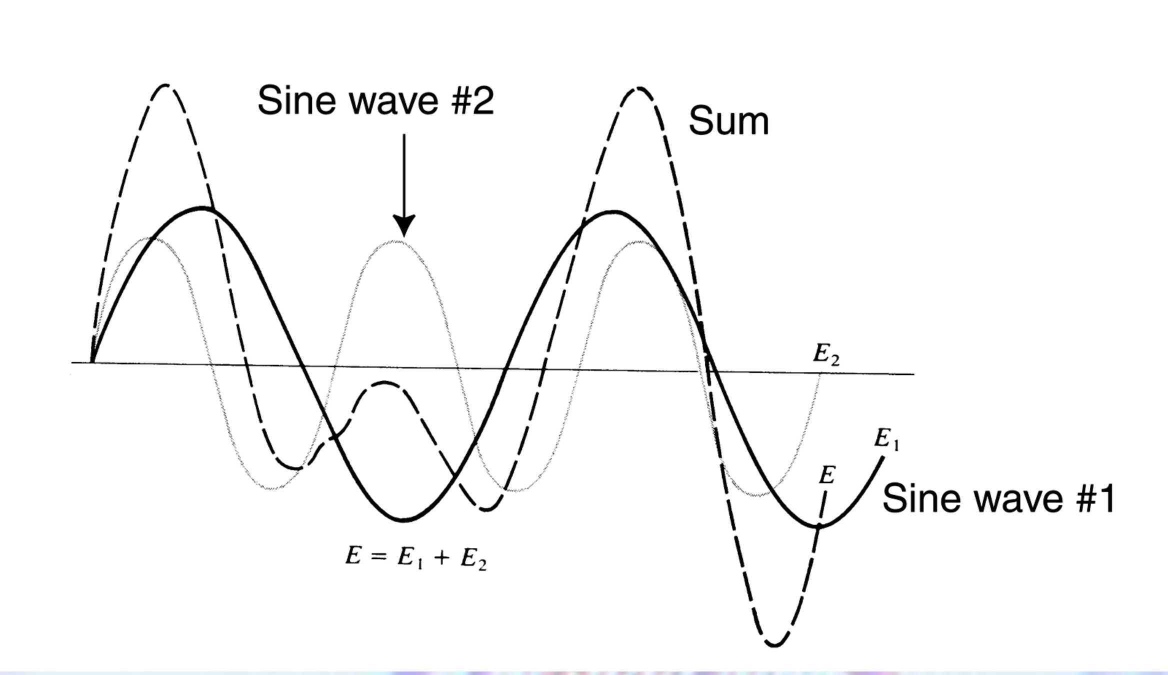
Joseph Fourier

Fourier was obsessed with the physics of heat and developed the Fourier series and transform to model heat-flow problems.

Joseph Fourier 1768 - 1830

Anharmonic waves are sums of sinusoids.

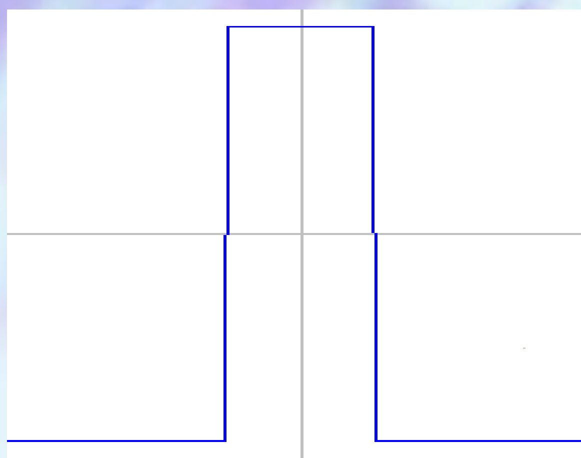
Consider the sum of two sine waves (i.e., harmonic waves) of different frequencies:



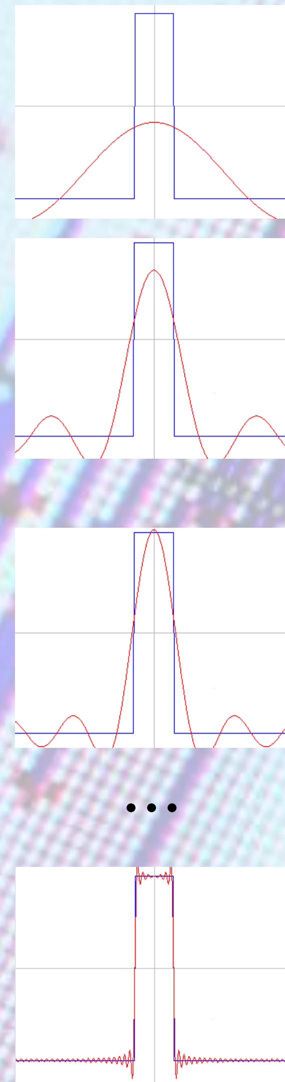
The resulting wave is periodic, but not harmonic.
Essentially all waves are anharmonic.

Introduction to Fourier Series

$f(x)$



$=$



$$a_0 + a_1 \cos(x)$$

$$\ldots + a_2 \cos(2x)$$

$$\ldots + a_3 \cos(3x)$$

$$\ldots$$

$$\ldots + a_n \cos(nx) + \ldots$$

Fourier series

- A Fourier series is a convenient representation of a periodic function.
- A Fourier series consists of a sum of sines and cosine terms.
- Sines and cosines are the most fundamental periodic functions.

Fourier series

- The formula for a Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{n=\infty}\left(a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right)\right)$$

Fourier series

- We have formulae for the coefficients (for the derivations see the course notes):

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$$

Fourier series - Orthogonality

- One very important property of sines and cosines is their orthogonality, expressed by:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin\left(\frac{2n\pi x}{T}\right)\sin\left(\frac{2m\pi x}{T}\right)dx = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2n\pi x}{T}\right)\cos\left(\frac{2m\pi x}{T}\right)dx = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2n\pi x}{T}\right)\sin\left(\frac{2m\pi x}{T}\right)dx = 0 \quad \text{for all } m, n$$

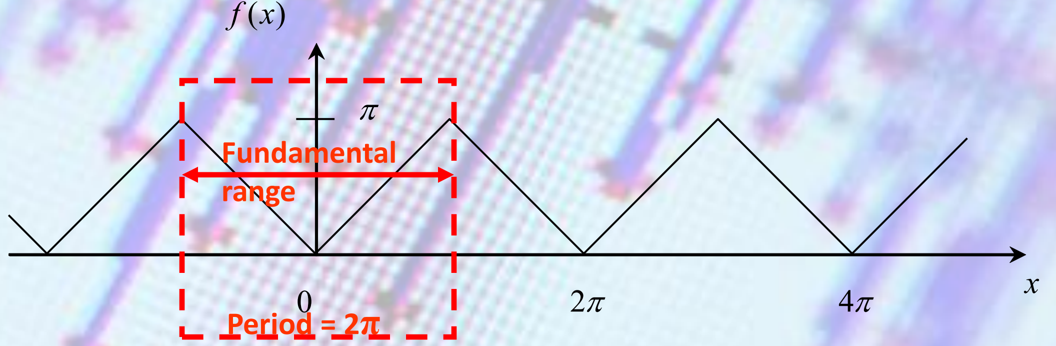
These formulae are used in the derivation of the formulae for a_n, b_n

Example – Fourier series

- Example – Find the coefficients for the Fourier series of:

$$f(x)=\begin{cases}-x & -\pi\le x\le 0\\ x & 0\le x\le \pi\end{cases}$$

$$f(x+2\pi)=f(x)$$



Example – Fourier series

- Find a_0

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$f(x)$ is an even function so:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \Rightarrow a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{0}^{\pi} xdx \Rightarrow a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \Rightarrow a_0 = \frac{\pi}{2}$$

Example – Fourier series

- Find a_n

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx \Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{2n\pi x}{2\pi}\right)dx$$

Since both functions are even their product is even:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \Rightarrow a_n = \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)dx$$

Example – Fourier series

- Find b_n

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx \quad \Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{2n\pi x}{2\pi}\right)dx$$

Since sine is an odd function and $f(x)$ is an even function, the product of the functions is odd:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \quad \Rightarrow b_n = 0$$

Example – Fourier series

- So we can put the coefficients back into the Fourier series formula:

$$f(x) = a_0 + \sum_{n=1}^{n=\infty} \left(a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right) \right)$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{n=\infty} \left(\frac{2}{\pi n^2} \left((-1)^n - 1\right) \cos(nx) \right)$$

$$\Rightarrow f(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos(x) + 0 - \frac{4}{9\pi} \cos(3x) + \ldots$$

Easy ways of finding Fourier coefficients

- There are some easy shortcuts for finding the Fourier coefficients.
- We can see that:

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx$$

is just the area under the fundamental range divided by the period.

Summary of finding coefficients

	function even	function odd	function neither
a_0	$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx = 0$ Though maybe easy to find using geometry	0	$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)dx = 0$ Though maybe easy to find using geometry
a_n	$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$	0	$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$
b_n	0	$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$	$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$

Week 9

Slide 158-176

Partial Sums

- The Fourier series gives the exact value of the function.
- However, it uses an infinite number of terms, so is impossible to calculate.
- We can evaluate the *partial sums* of a Fourier series by only evaluating a set number of the terms.

Partial Sums

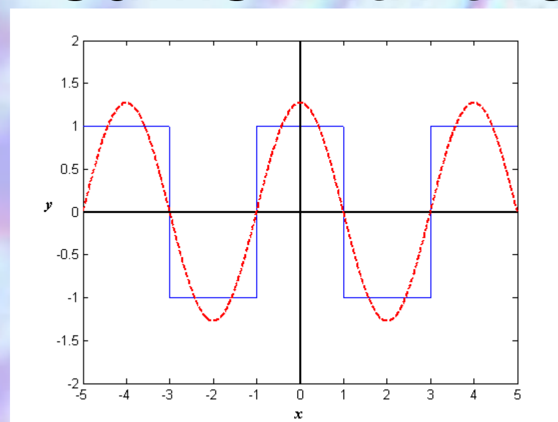
- For partial sums we use the notation:

$$S_N(x) = a_0 + \sum_{n=1}^{n=N} \left(a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right) \right)$$

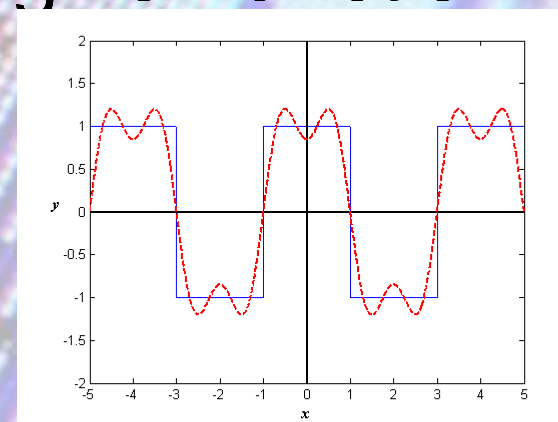
To represent a partial sum with N terms.

Example 1 – Partial Sums

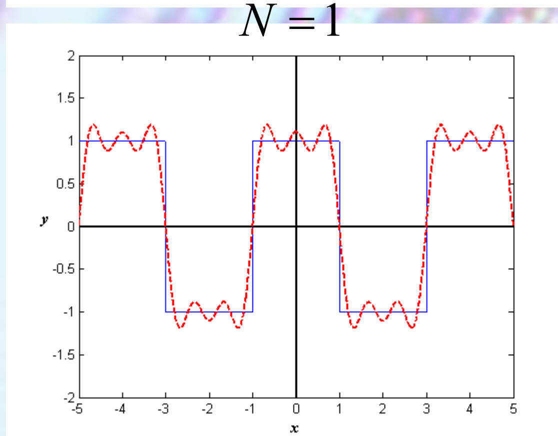
- Compare the plots of the partial sums with the original function:



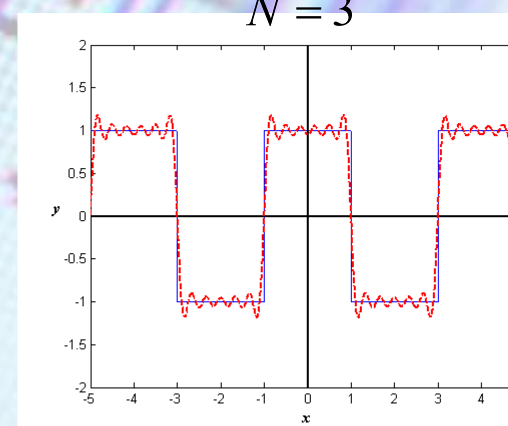
$N = 1$



$N = 3$



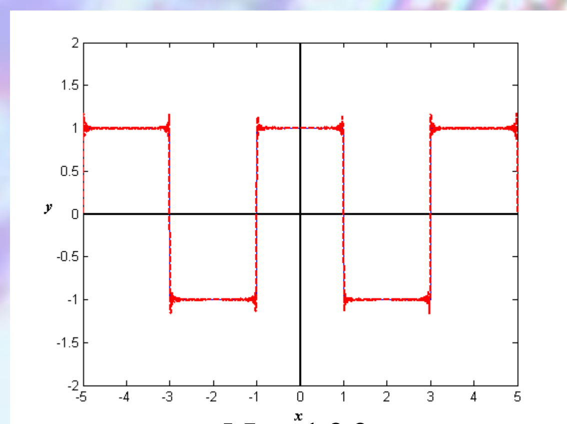
$N = 5$



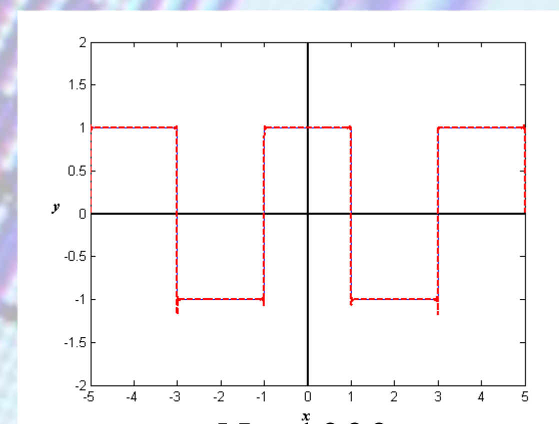
$N = 11$

Example 1 – Partial Sums

- Compare the plots of the partial sums with the original function:



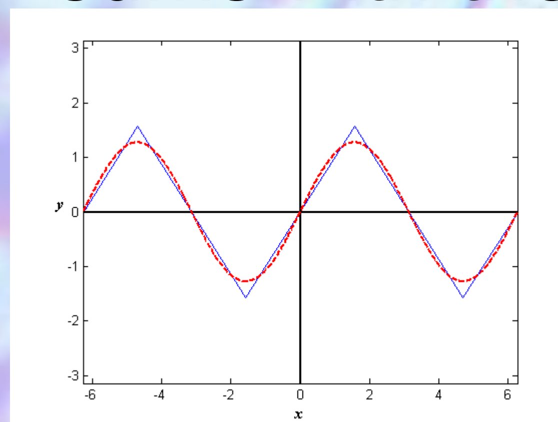
$N = 100$



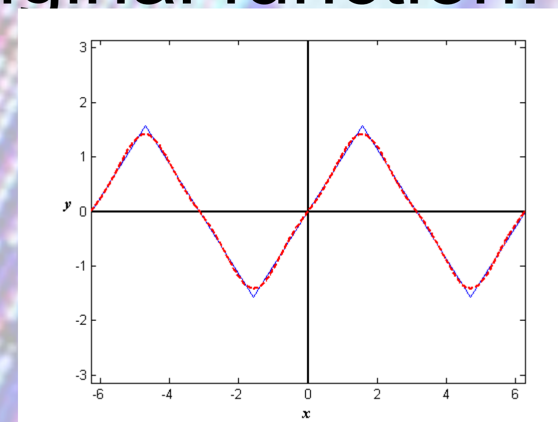
$N = 1000$

Example 1 – Partial Sums

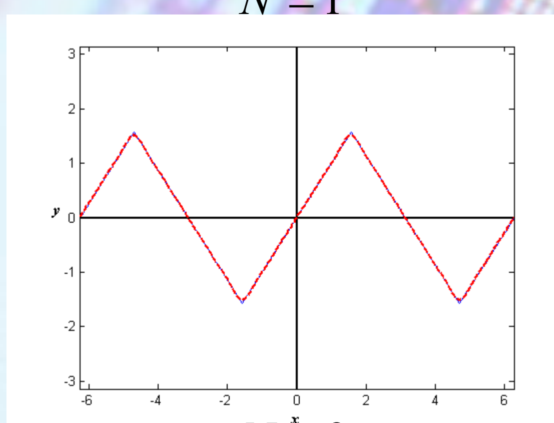
- Compare the plots of the partial sums with the original function:



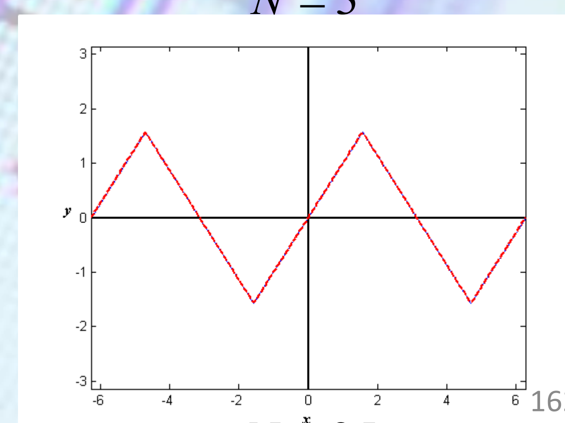
$N = 1$



$N = 3$



$N = 9$



$N = 25$

The Fourier Transform

Consider the Fourier coefficients. Let's define a function $F(m)$ that incorporates both cosine and sine series coefficients, with the sine series distinguished by making it the imaginary component:

$$F(m) \equiv F_m - i\,F'_m = \int f(t)\cos(mt)\,dt \;-\; i\int f(t)\sin(mt)\,dt$$

Let's now allow $f(t)$ to range from $-\infty$ to ∞, so we'll have to integrate from $-\infty$ to ∞, and let's redefine m to be the "frequency," which we'll now call ω:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

The Fourier Transform

$F(\omega)$ is called the Fourier Transform of $f(t)$. **It contains equivalent information to that in $f(t)$.** We say that $f(t)$ lives in the **time domain**, and $F(\omega)$ lives in the **frequency domain**. $F(\omega)$ is just another way of looking at a function or wave.

The Inverse Fourier Transform

The Fourier Transform takes us from $f(t)$ to $F(\omega)$.
How about going back?

Recall our formula for the Fourier Series of $f(t)$:

$$f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

Now transform the sums to integrals from –∞ to ∞, and again replace F_m with $F(\omega)$. Remembering the fact that we introduced a factor of i (and including a factor of 2 that just crops up), we have:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega$$

Inverse Fourier Transform

The Fourier Transform and its Inverse

The Fourier Transform and its Inverse:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt$$

FourierTransform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t)\, d\omega$$

Inverse Fourier Transform

So we can transform to the frequency domain and back. Interestingly, these transformations are very similar.

There are different definitions of these transforms. The 2π can occur in several places, but the idea is generally the same.

Fourier Transform Notation

There are several ways to denote the Fourier transform of a function.

If the function is labeled by a lower-case letter, such as f, we can write:

$$f(t) \rightarrow F(\omega)$$

If the function is already labeled by an upper-case letter, such as E, we can write:

$$E(t) \rightarrow \mathscr{F}\{E(t)\}$$

or:

$$E(t) \rightarrow \tilde{E}(\omega)$$

- Sometimes, this symbol is used instead of the arrow: $\supset$

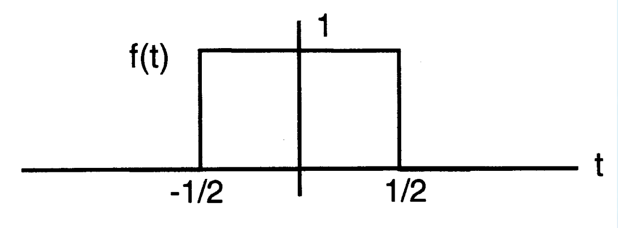
Example: the Fourier Transform of a rectangle function: rect(t)

$$F(\omega) = \int_{-1/2}^{1/2} \exp(-i\omega t)\,dt = \frac{1}{-i\omega}\left[\exp(-i\omega t)\right]_{-1/2}^{1/2}$$

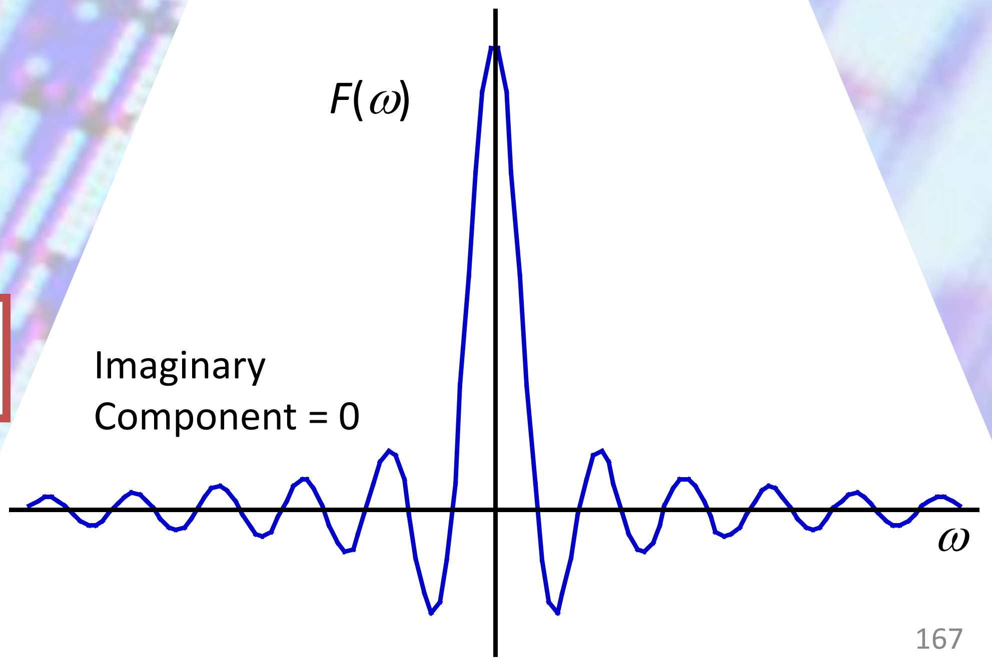
$$= \frac{1}{-i\omega}\left[\exp(-i\omega/2) - \exp(i\omega/2)\right]$$

$$= \frac{1}{(\omega/2)} \frac{\exp(i\omega/2) - \exp(-i\omega/2)}{2i}$$

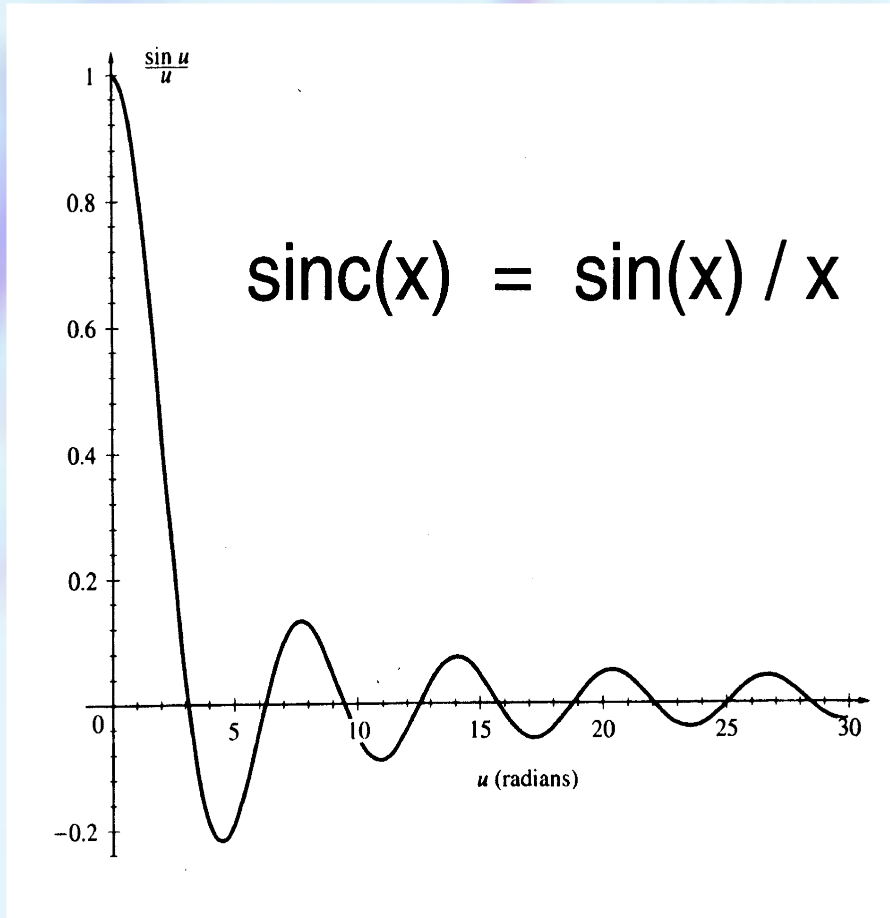
$$= \frac{\sin(\omega/2)}{(\omega/2)}$$



$$F(\omega) = \mathrm{sinc}(\omega/2)$$



Sinc(x) and why it's important



Sinc($x/2$) is the Fourier transform of a rectangle function.

Sinc^2($x/2$) is the Fourier transform of a triangle function.

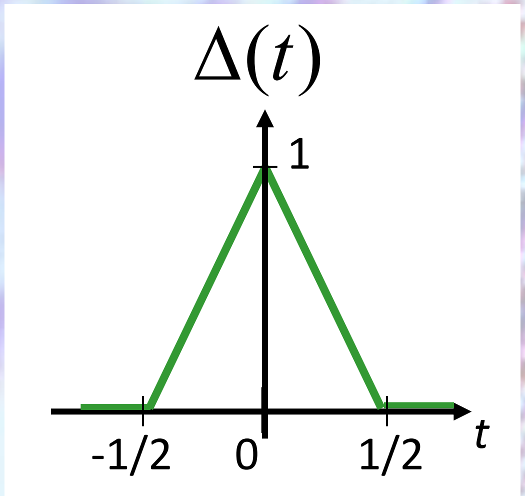
Sinc^2(ax) is the diffraction pattern from a slit.

It just crops up everywhere...

The Fourier Transform of the triangle function, Δ(t), is sinc²(ω/2)

The triangle function is just what it sounds like.

Sometimes people use Λ(*t*), too, for the triangle function.



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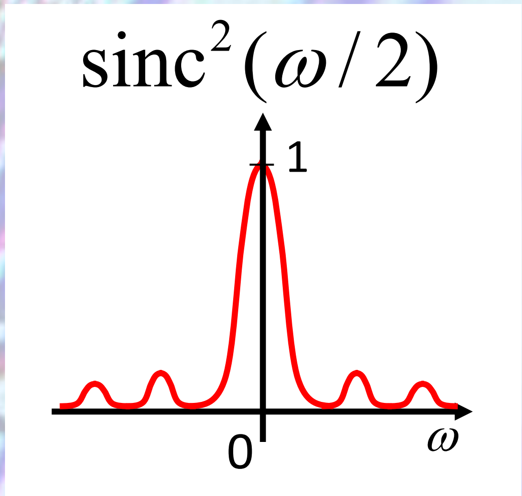


We'll prove this when we learn about convolution.

Example: the Fourier Transform of a decaying exponential: exp(-*at*) (*t* > 0)

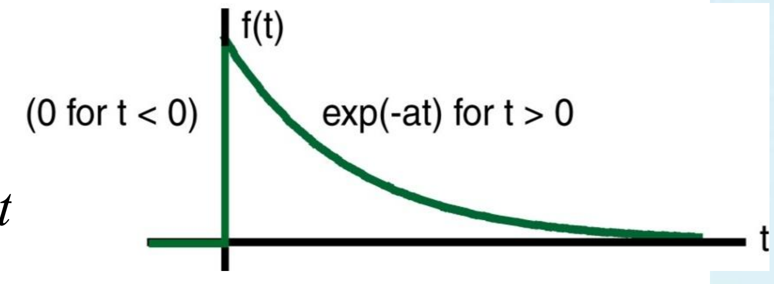
$$F(\omega) = \int_0^\infty \exp(-at)\exp(-i\omega t)\,dt$$

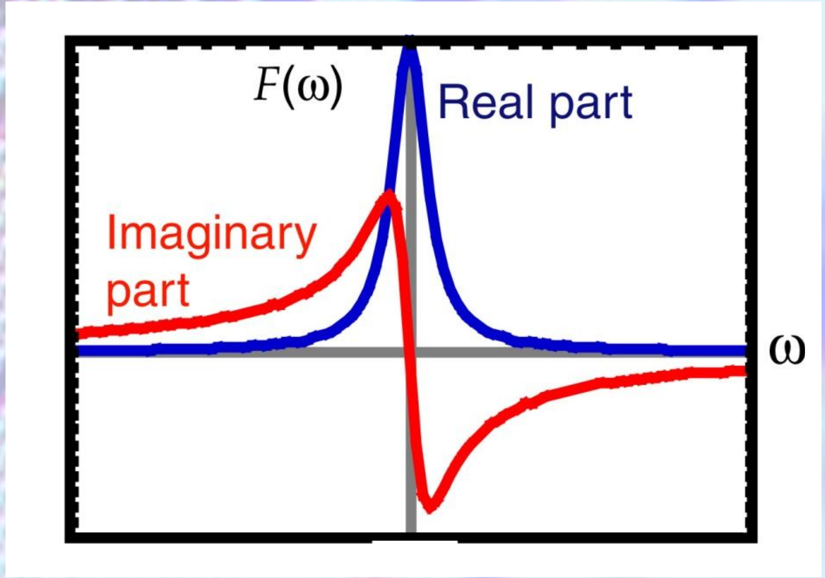
$$= \int_0^\infty \exp(-at - i\omega t)\,dt = \int_0^\infty \exp(-[a+i\omega]t)\,dt$$

$$= \frac{-1}{a+i\omega}\exp(-[a+i\omega]t)\Big|_0^{+\infty} = \frac{-1}{a+i\omega}[\exp(-\infty)-\exp(0)]$$

$$= \frac{-1}{a+i\omega}[0-1]$$

$$= \frac{1}{a+i\omega}$$

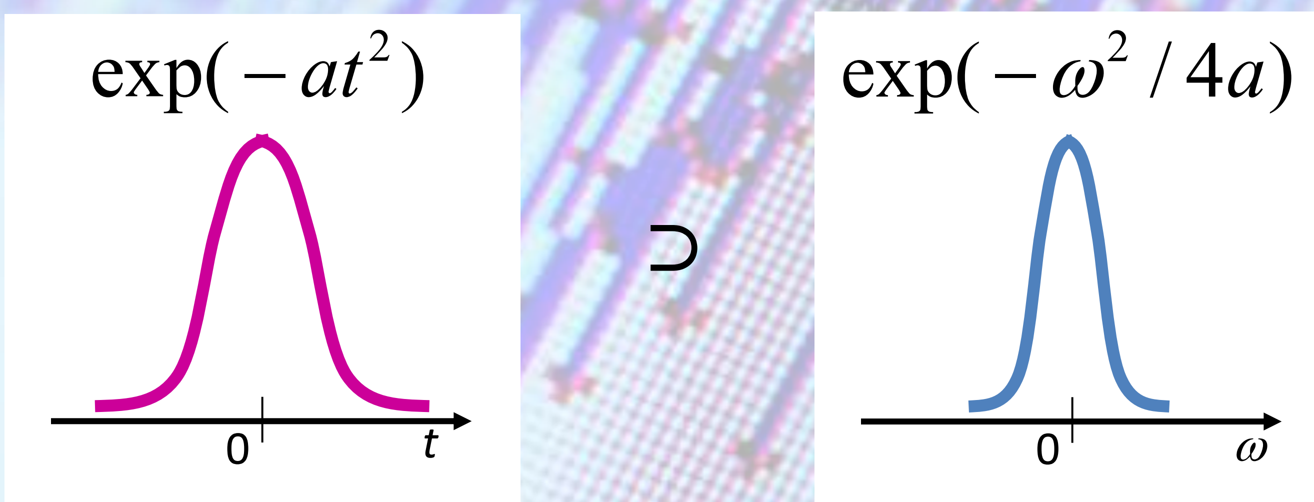




Example: the Fourier Transform of a Gaussian, $\exp(-at^2)$, is itself!

$$\mathcal{F}\{\exp(-at^2)\} = \int_{-\infty}^{\infty} \exp(-at^2)\exp(-i\omega t)\,dt$$

$$\propto \exp(-\omega^2 / 4a)$$ The details are a HW problem!

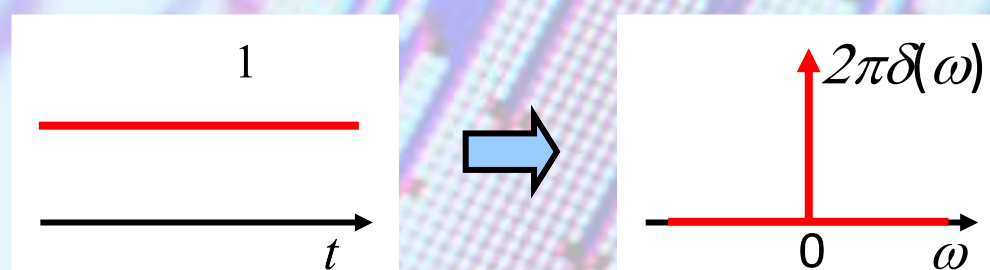


The Fourier Transform of $\delta(t)$ is 1.

$$\int_{-\infty}^{\infty} \delta(t)\exp(-i\omega t)\,dt = \exp(-i\omega[0]) = 1$$

And the Fourier Transform of 1 is $2\pi\delta(\omega)$:

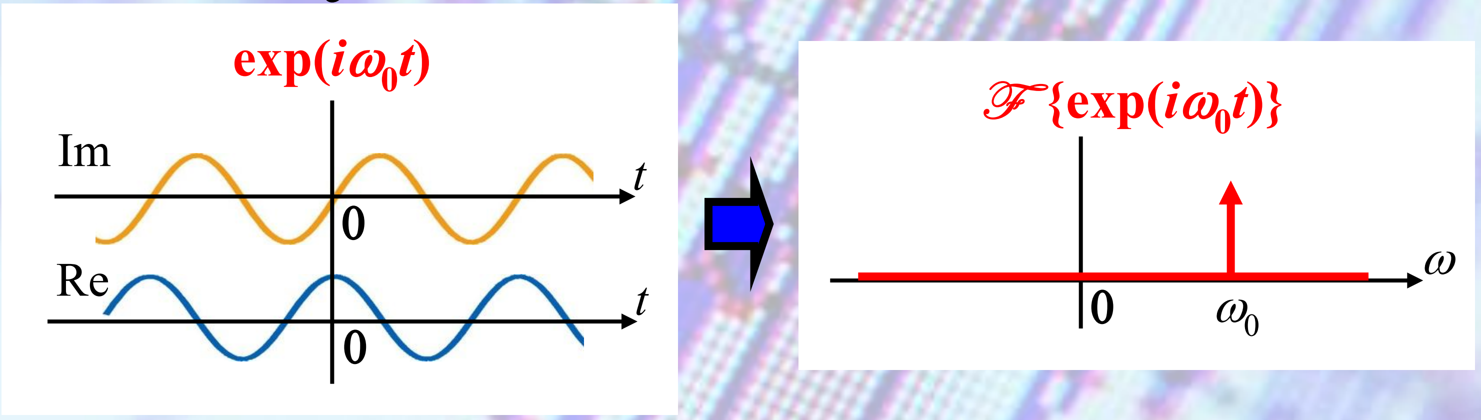
$$\int_{-\infty}^{\infty} 1\exp(-i\omega t)\,dt = 2\pi\;\delta(\omega)$$



The Fourier transform of $\exp(i\omega_0 t)$

$$\mathcal{F}\{\exp(i\omega_0 t)\} = \int_{-\infty}^{\infty} \exp(i\omega_0 t)\exp(-i\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt = 2\pi\,\delta(\omega-\omega_0)$$



The function $\exp(i\omega_0 t)$ is the essential component of Fourier analysis. It is a pure frequency.

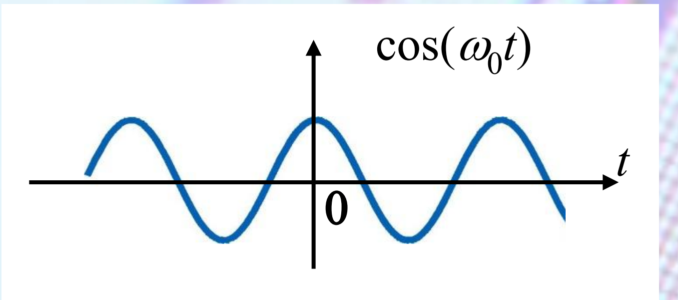
The Fourier transform of $\cos(\omega_0 t)$

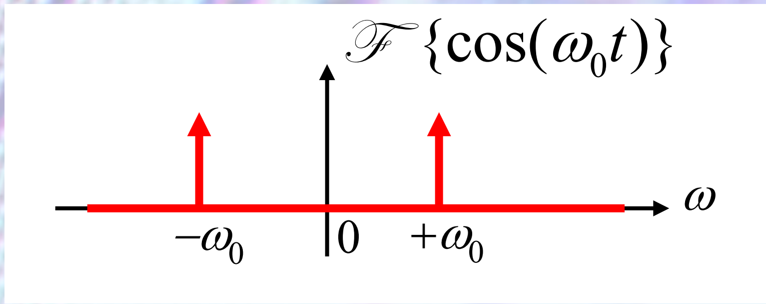
$$\mathcal{F}\{\cos(\omega_0 t)\} = \int_{-\infty}^{\infty} \cos(\omega_0 t) \exp(-i\,\omega t)\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(i\,\omega_0\, t) + \exp(-i\,\omega_0\, t)\right] \exp(-i\,\omega t)\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\, dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega+\omega_0]t)\, dt$$

$$= \pi\,\delta(\omega-\omega_0) \quad + \quad \pi\,\delta(\omega+\omega_0)$$





Fourier Transform Symmetry Properties

Expanding the Fourier transform of a function, $f(t)$:

$$F(\omega) = \int_{-\infty}^{\infty} [\mathrm{Re}\{f(t)\} + i\,\mathrm{Im}\{f(t)\}]\;[\cos(\omega t) - i\sin(\omega t)]\,dt$$

Expanding more, noting that: $\int_{-\infty}^{\infty} O(t)\,dt = 0$ if $O(t)$ is an odd function

= 0 if $\mathrm{Re}\{f(t)\}$ is odd ↓   = 0 if $\mathrm{Im}\{f(t)\}$ is even ↓

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\cos(\omega t)\,dt + \int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\sin(\omega t)\,dt \quad \leftarrow \mathrm{Re}\{F(\omega)\}$$

= 0 if $\mathrm{Im}\{f(t)\}$ is odd ↓   = 0 if $\mathrm{Re}\{f(t)\}$ is even ↓

$$+\; i\int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\cos(\omega t)\,dt - i\int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\sin(\omega t)\,dt \quad \leftarrow \mathrm{Im}\{F(\omega)\}$$

Even functions of ω Odd functions of ω

Some functions don't have Fourier transforms.

The condition for the existence of a given $F(\omega)$ is:

$$\int_{-\infty}^{\infty} \left| f(t) \right| \, dt \;<\; \infty$$

Functions that do not asymptote to zero in both the $+\infty$ and $-\infty$ directions generally do not have Fourier transforms.

So we'll assume that all functions of interest go to zero at $\pm\infty$.

Week 10

Slide 178-206

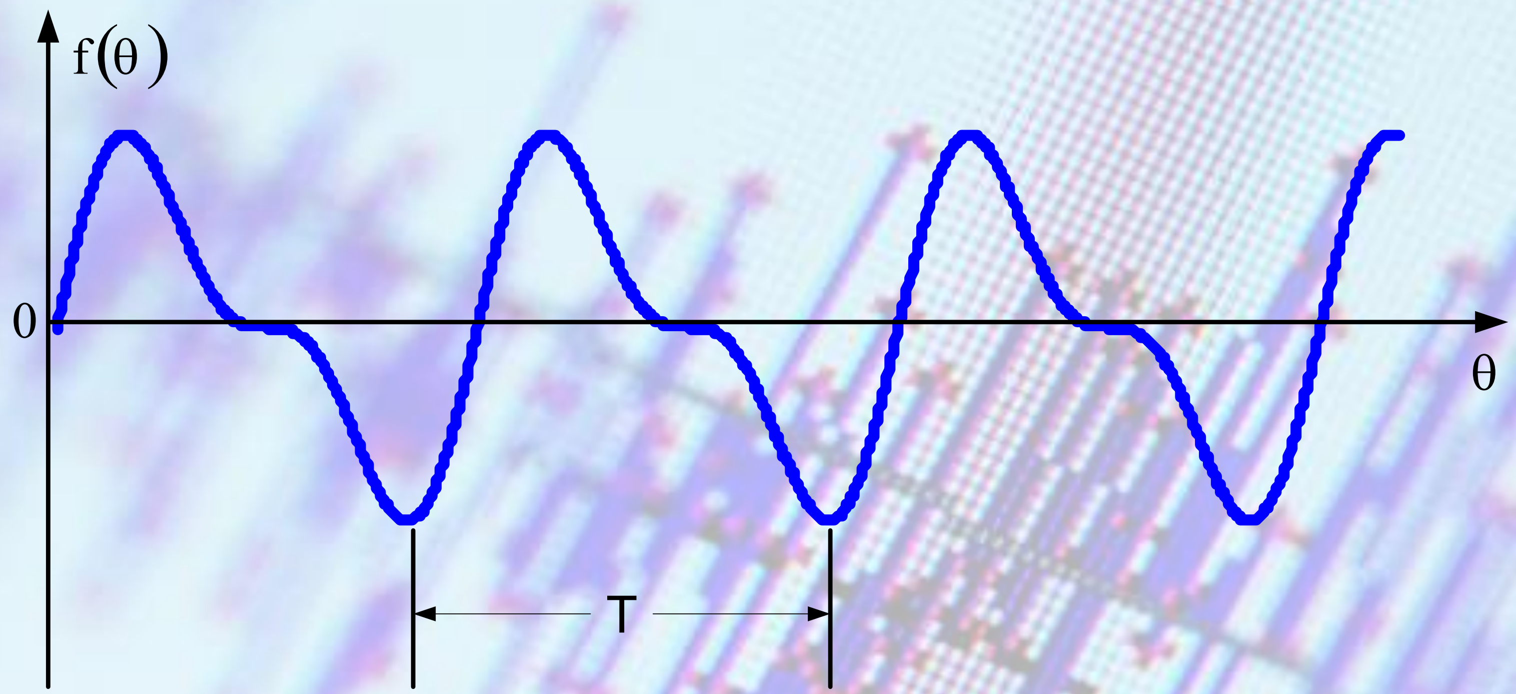
Periodic Functions

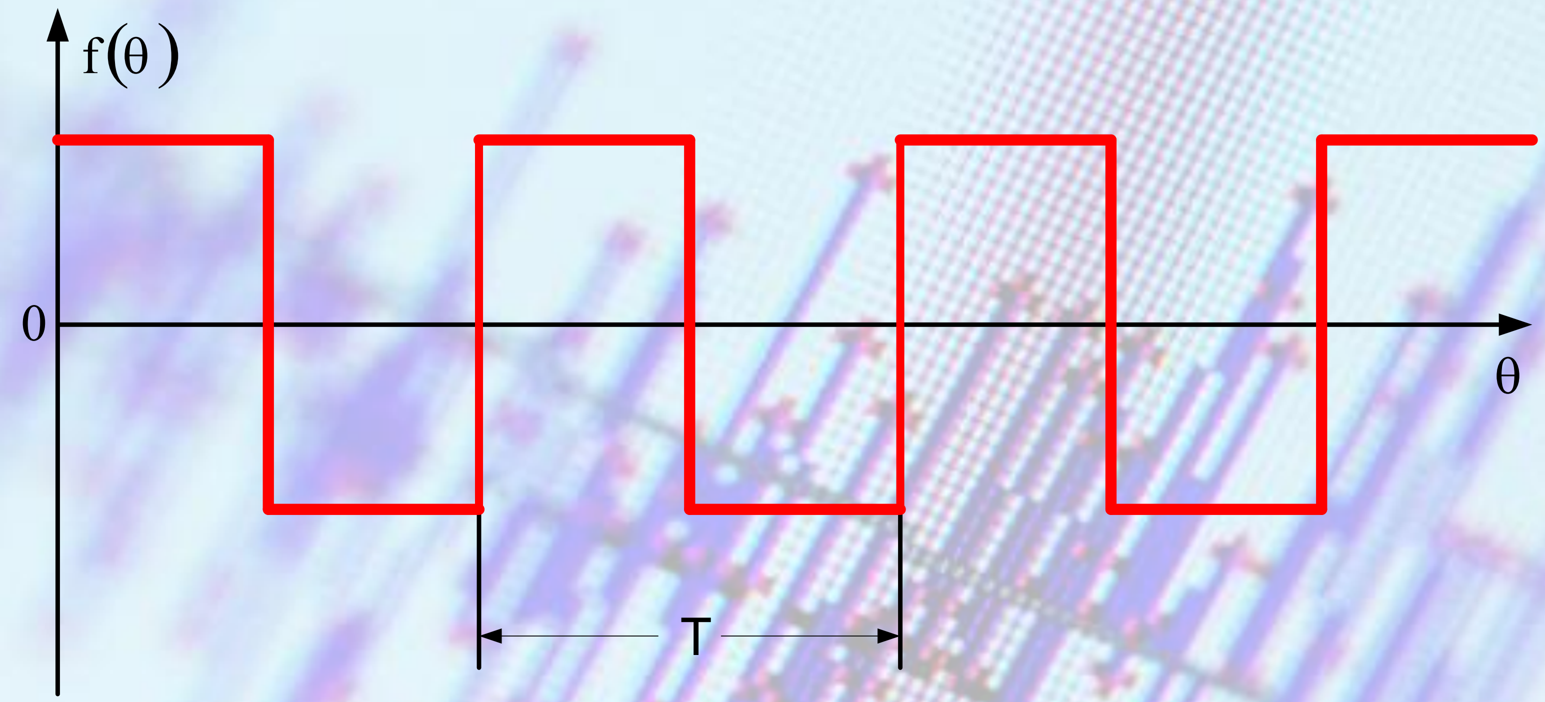
A function $f(\theta)$ is periodic

if it is defined for all real θ

and if there is some positive number,

T such that $f(\theta + T) = f(\theta)$.


f(θ)
0
θ
T


f(θ)
0
θ
T


f(θ)
0
θ
T

Fourier Series

$f(\theta)$ be a periodic function with period 2π

The function can be represented by a trigonometric series as:

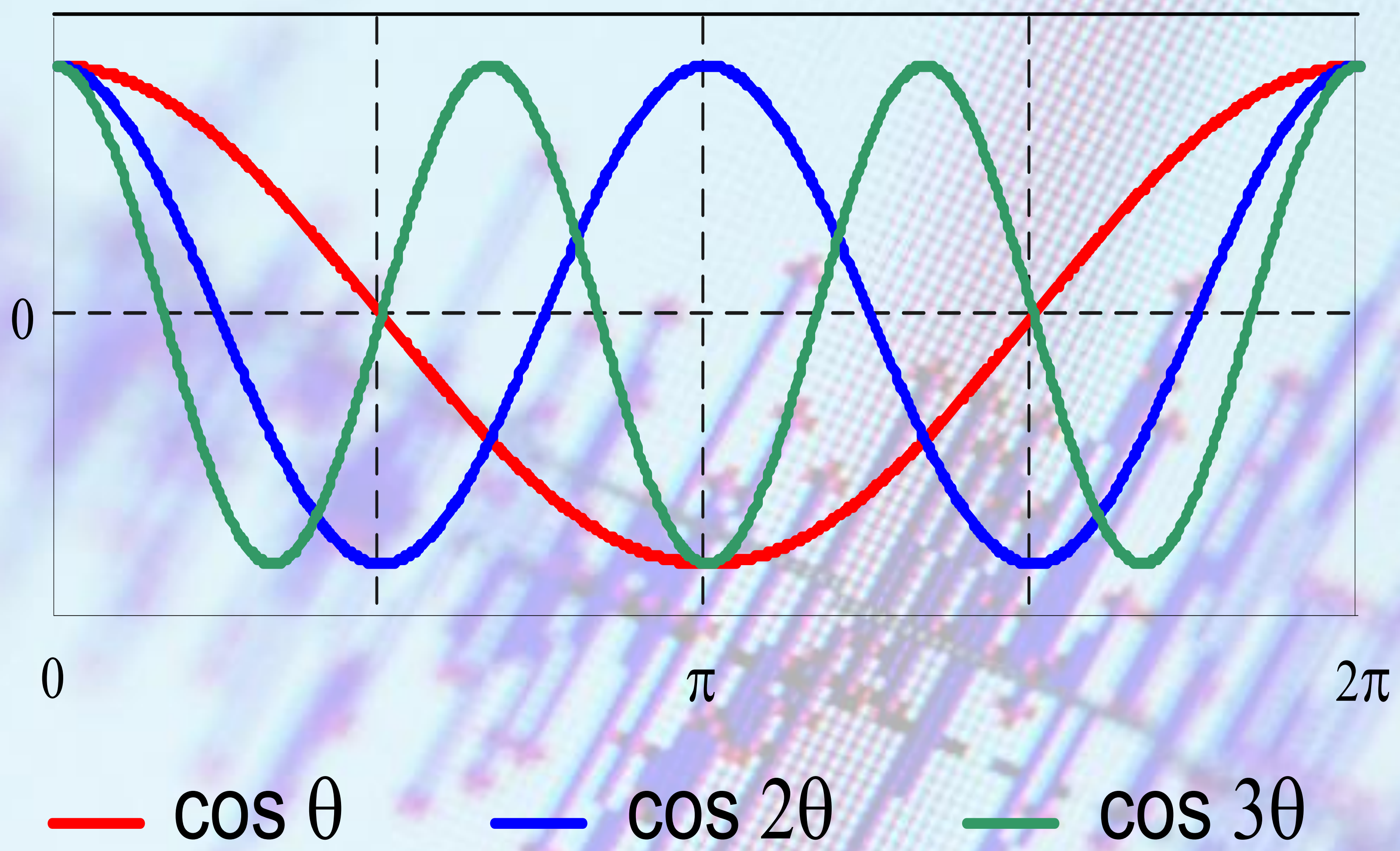
$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

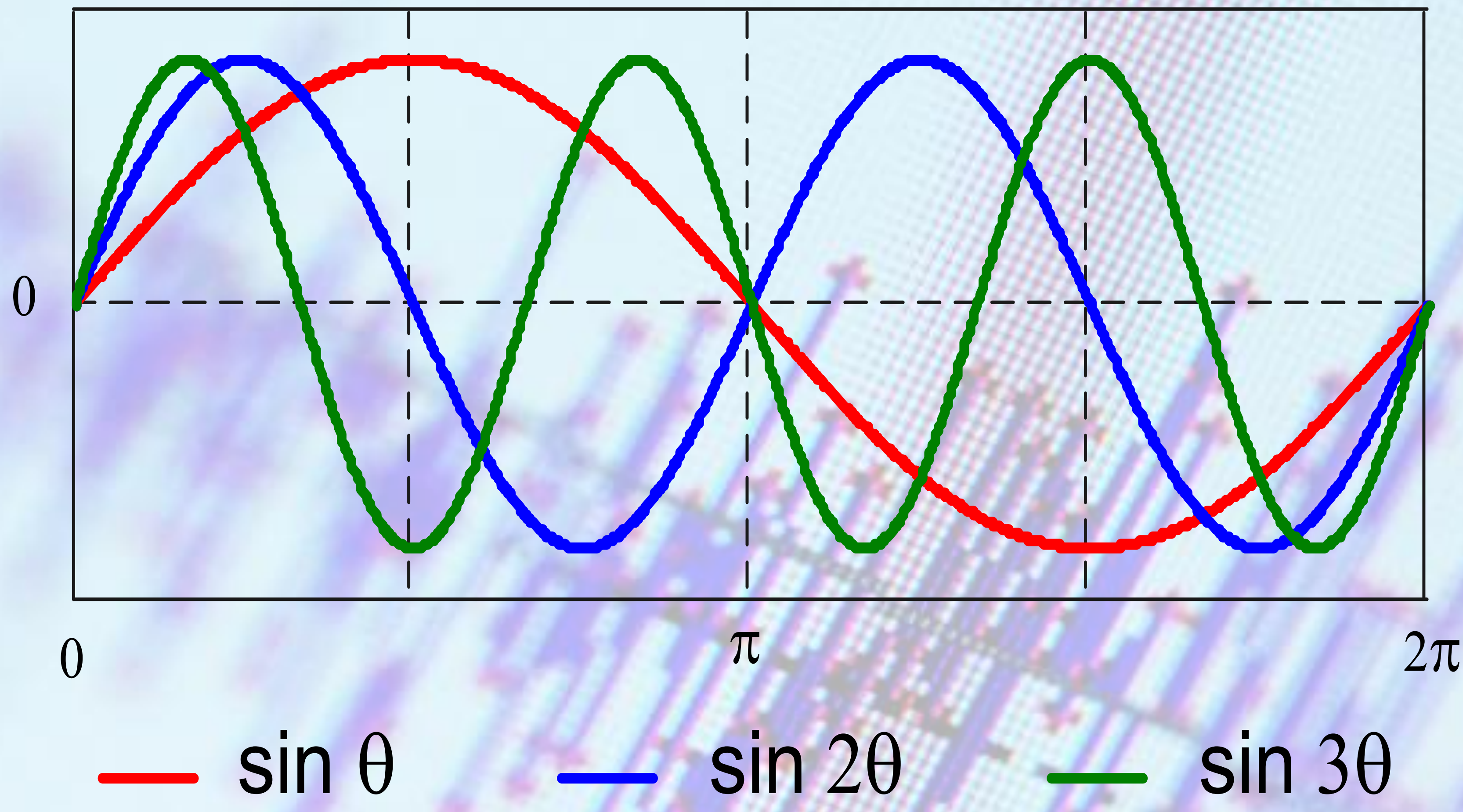
$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

What kind of trigonometric (series) functions are we talking about?

$$cos\,\theta,\ cos\,2\theta,\ cos\,3\theta \cdots \quad \textbf{and}$$

$$sin\,\theta,\ sin\,2\theta,\ sin\,3\theta \cdots$$


0
0
π
2π
cos θ
cos 2θ
cos 3θ


0
0
π
2π
sin θ
sin 2θ
sin 3θ

We want to determine the coefficients,

a_n and b_n .

Let us first remember some useful integrations.

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \pi \qquad n = m$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} sin\, \boldsymbol{n}\theta\, cos\, \boldsymbol{m}\theta \boldsymbol{d}\theta = \mathbf{0}$$

for all values of *m*.

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\theta \, d\theta - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} sin\, n\theta\, sin\, m\theta d\theta = 0 \quad n \neq m$$

$$\int_{-\pi}^{\pi} sin\, n\theta\, sin\, m\theta d\theta = \pi \quad n = m$$

Determine a_0

Integrate both sides of (1) from

$-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\right] d\theta$$

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi} a_0\,d\theta + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} a_n \cos n\theta \right) d\theta$$

$$+ \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} b_n \sin n\theta \right) d\theta$$

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta = \int_{-\pi}^{\pi} a_0\,d\theta + 0 + 0$$

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

a_0 is the average (dc) value of the function, $f(\theta)$.

You may integrate both sides of (1) from

$\mathbf{0}$ to $\mathbf{2\pi}$ instead.

$$\int_0^{2\pi} f(\theta)\,d\theta$$

$$= \int_0^{2\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\right] d\theta$$

It is alright as long as the integration is performed over one period.

$$\int_0^{2\pi} f(\theta)\,d\theta$$

$$= \int_0^{2\pi} a_0\,d\theta + \int_0^{2\pi}\left(\sum_{n=1}^{\infty} a_n \cos n\theta\right)d\theta$$

$$+ \int_0^{2\pi}\left(\sum_{n=1}^{\infty} b_n \sin n\theta\right)d\theta$$

$$\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{2\pi} a_0\,d\theta + 0 + 0$$

$$\int_0^{2\pi} f(\theta)\,d\theta = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

Determine a_n

Multiply (1) by $\cos m\theta$

and then Integrate both sides from

$-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\cos m\theta \, d\theta$$

$$= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \right] \cos m\theta \, d\theta$$

Let us do the integration on the right-hand-side one term at a time.

First term,

$$\int_{-\pi}^{\pi} a_0 \cos m\theta \, d\theta = 0$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \cos m\theta \, d\theta$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \cos m\theta \, d\theta = a_m \pi$$

Third term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin n\theta \cos m\theta \, d\theta = 0$$

Therefore,

$$\int_{-\pi}^{\pi} f(\theta)\cos m\theta\, d\theta = a_m \pi$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\theta\, d\theta \qquad m = 1, 2, \cdots$$

Determine b_n

Multiply (1) by $\sin m\theta$

and then Integrate both sides from

$-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\sin m\theta \, d\theta$$

$$= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\right]\sin m\theta \, d\theta$$

Let us do the integration on the right-hand-side one term at a time.

First term,

$$\int_{-\pi}^{\pi} a_0 \sin m\theta \, d\theta = 0$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \sin m\theta \, d\theta$$

Second term,

$$\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos n\theta \sin m\theta d\theta = 0$$

Third term,

$$\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n \sin n\theta \sin m\theta\, d\theta = b_m\pi$$

Therefore,

$$\int_{-\pi}^{\pi} f(\theta)\sin m\theta d\theta = b_m \pi$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin m\theta d\theta \qquad m = 1, 2, \cdots$$

The coefficients are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\theta\,d\theta \qquad m = 1, 2, \cdots$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin m\theta\,d\theta \qquad m = 1, 2, \cdots$$

We can write *n* in place of *m*:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta \qquad n = 1, 2, \cdots$$

The integrations can be performed from

0 to 2π instead.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

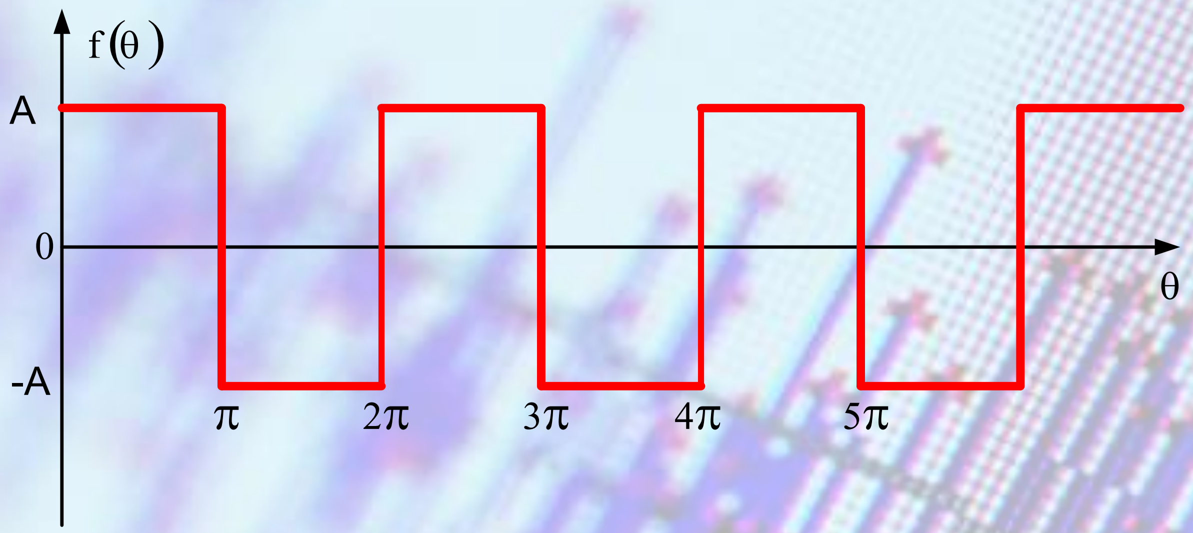
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \quad n = 1, 2, \cdots$$

Week 11

Slide 208-232

Example 1. Find the Fourier series of the following periodic function.



$$f(\theta) = A \quad \textit{when} \quad 0 < \theta < \pi$$

$$= -A \quad \textit{when} \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} f(\theta)\,d\theta + \int_{\pi}^{2\pi} f(\theta)\,d\theta\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,d\theta + \int_{\pi}^{2\pi} -A\,d\theta\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\theta\, d\theta + \int_{\pi}^{2\pi}(-A)\cos n\theta\, d\theta\right]$$

$$= \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[-A\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\sin n\theta \, d\theta + \int_{\pi}^{2\pi}(-A)\sin n\theta \, d\theta\right]$$

$$= \frac{1}{\pi}\left[-A\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[1+1+1+1\right]$$

$$= \frac{4A}{n\pi} \quad \textbf{when n is odd}$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[-1 + 1 + 1 - 1\right]$$

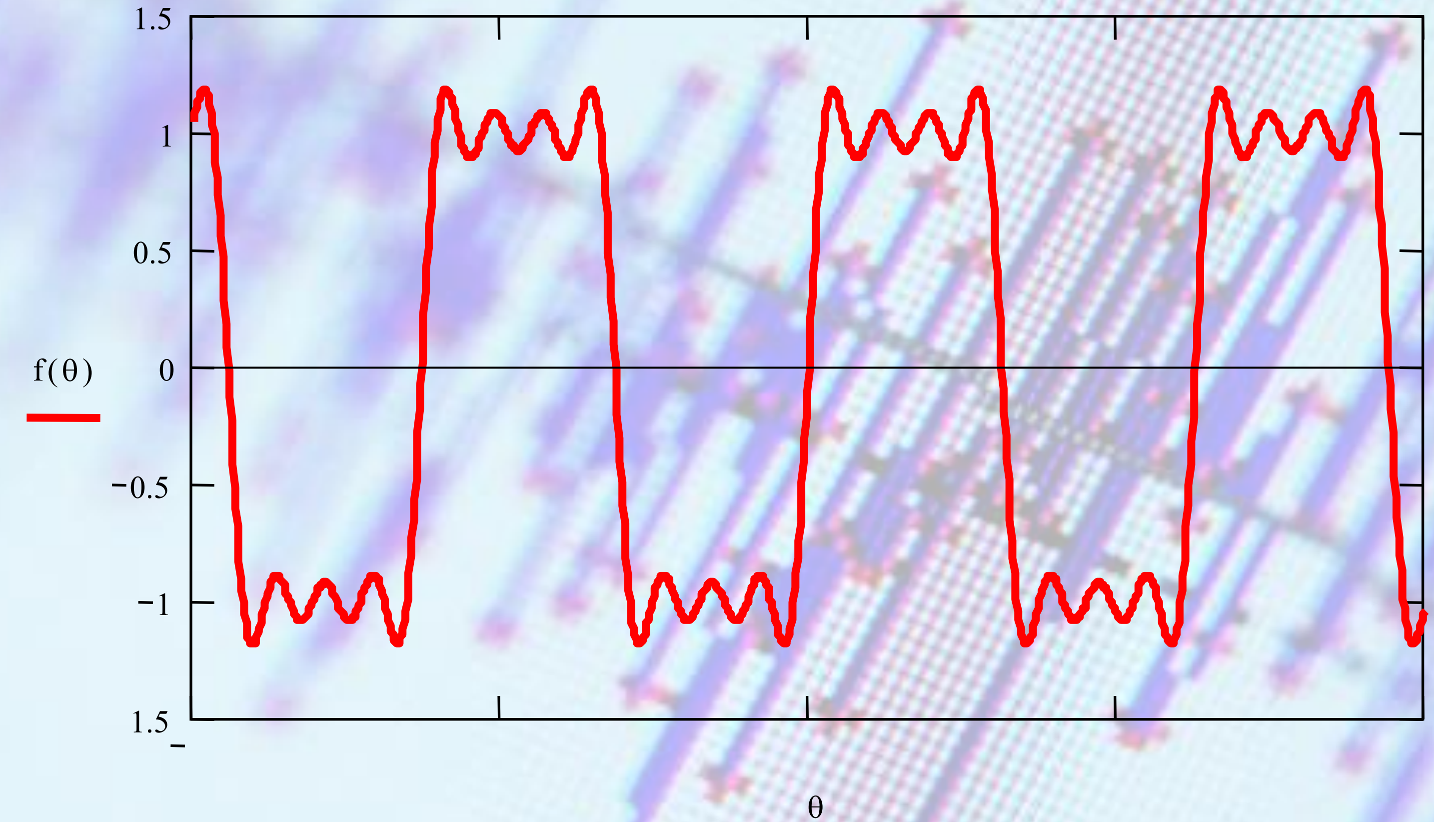
$$= 0 \quad \text{when n is even}$$

Therefore, the corresponding Fourier series is

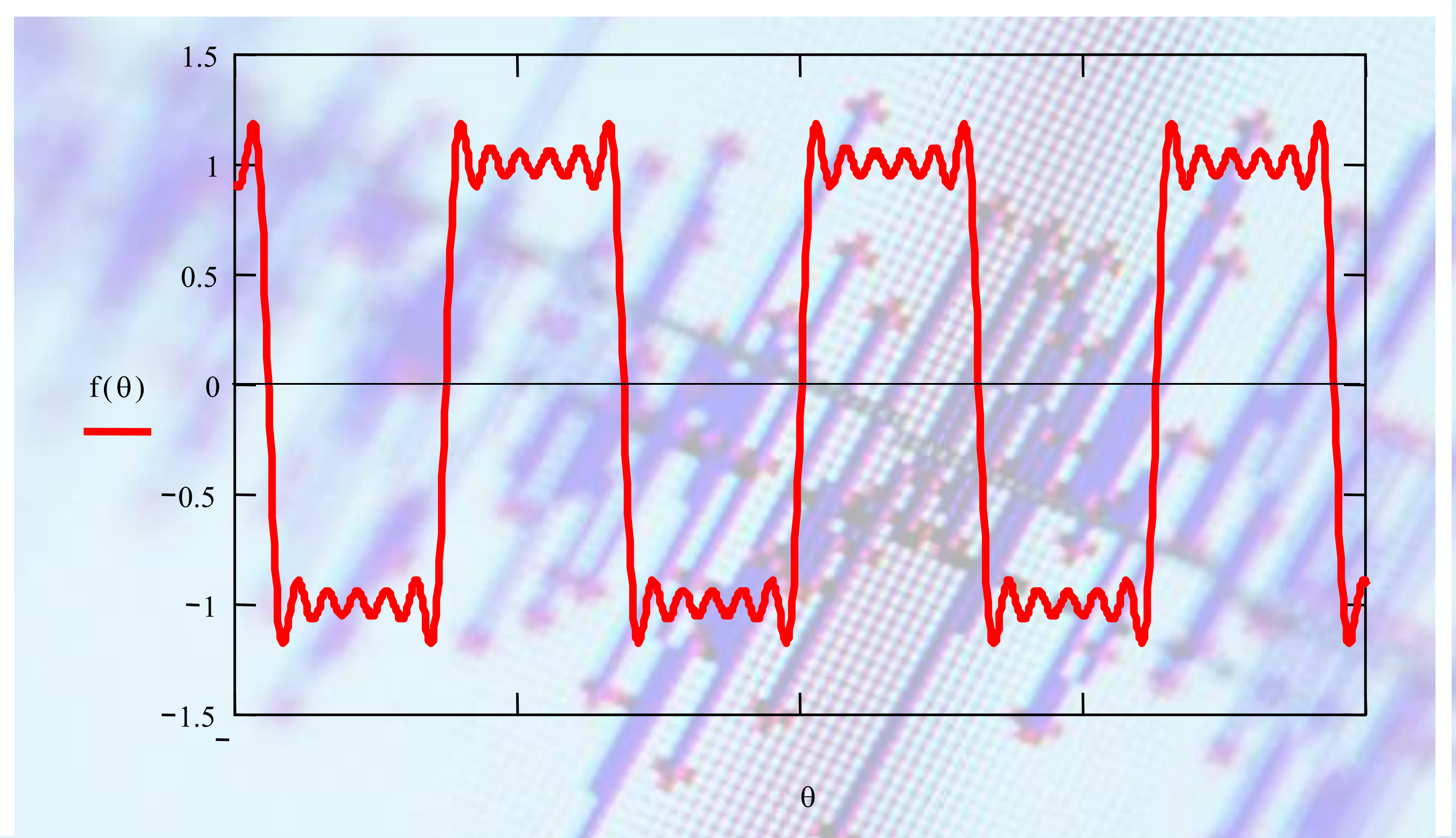
$$\frac{4A}{\pi}\left(\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta+\frac{1}{7}\sin 7\theta+\cdots\right)$$

In writing the Fourier series we may not be able to consider infinite number of terms for practical reasons. The question therefore, is – how many terms to consider?

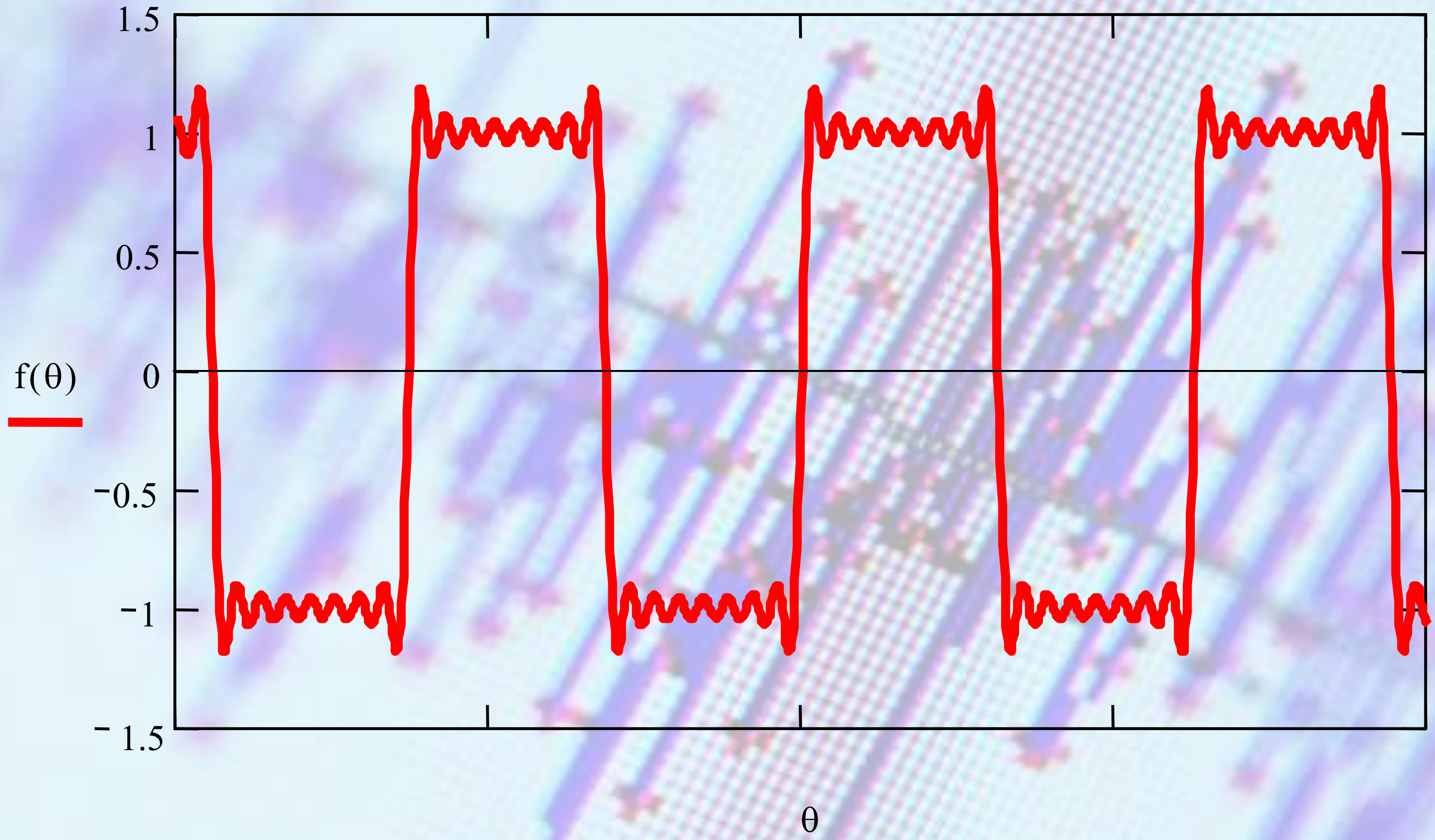
When we consider 4 terms as shown in the previous slide, the function looks like the following.



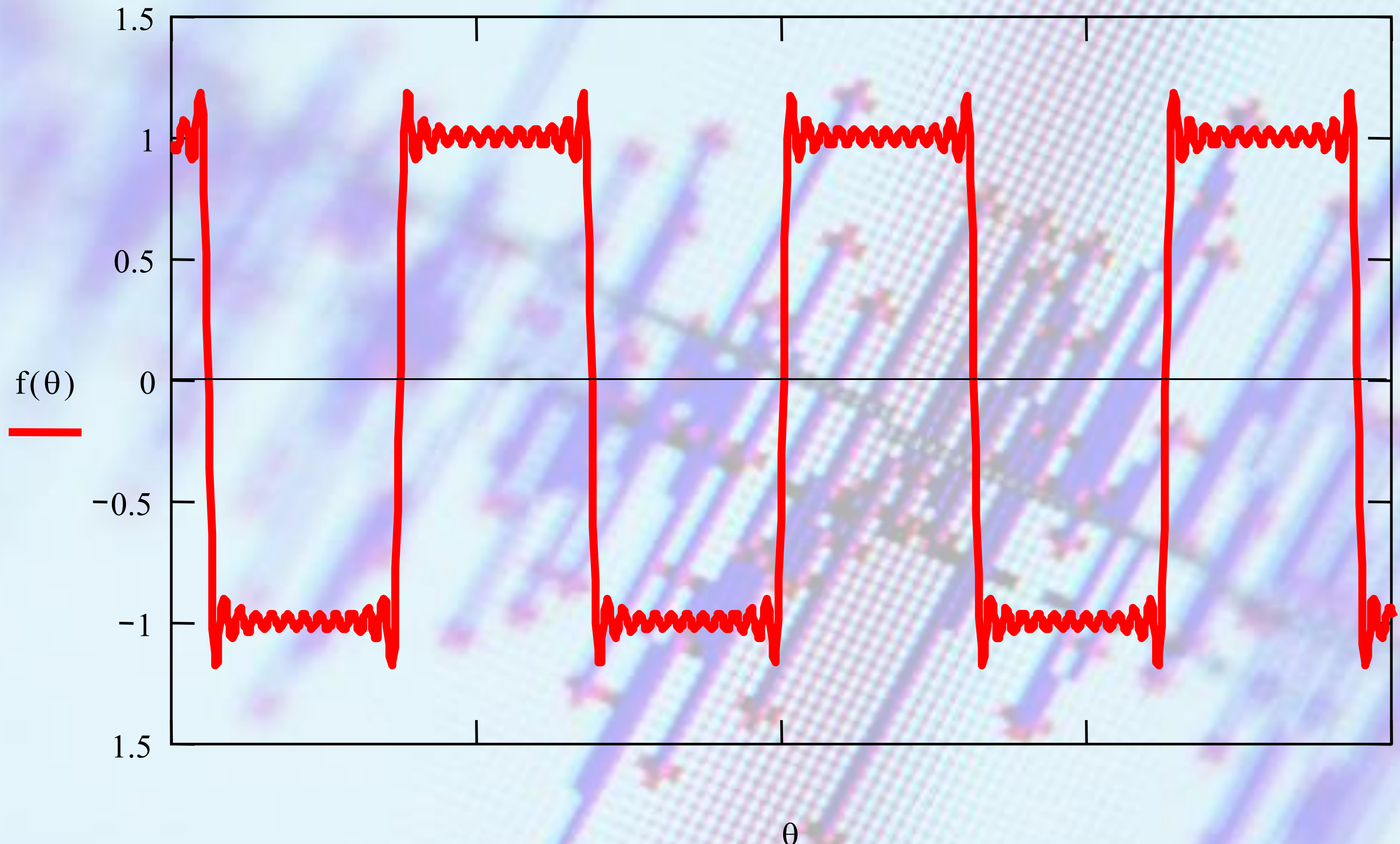
When we consider 6 terms, the function looks like the following.



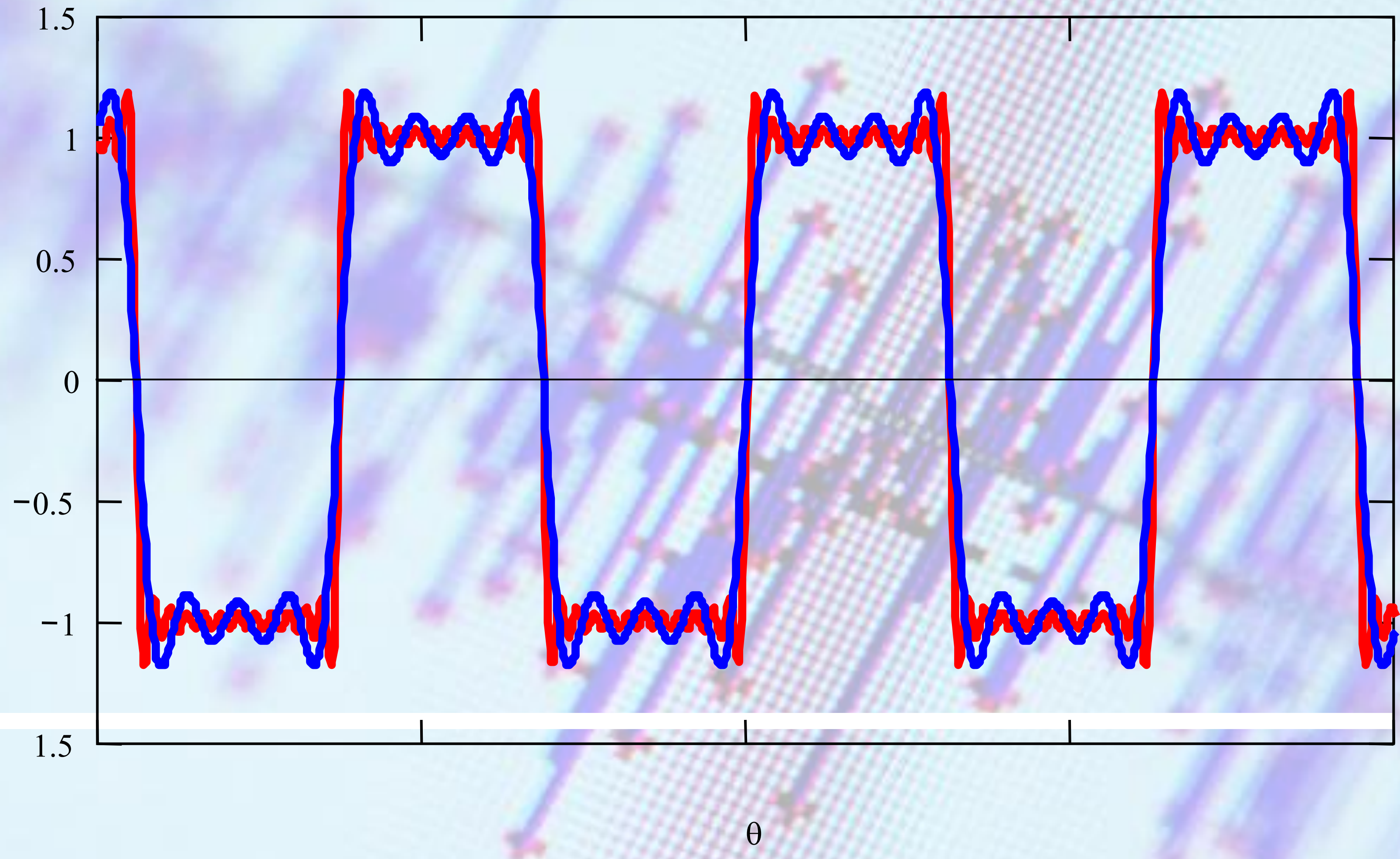
When we consider 8 terms, the function looks like the following.



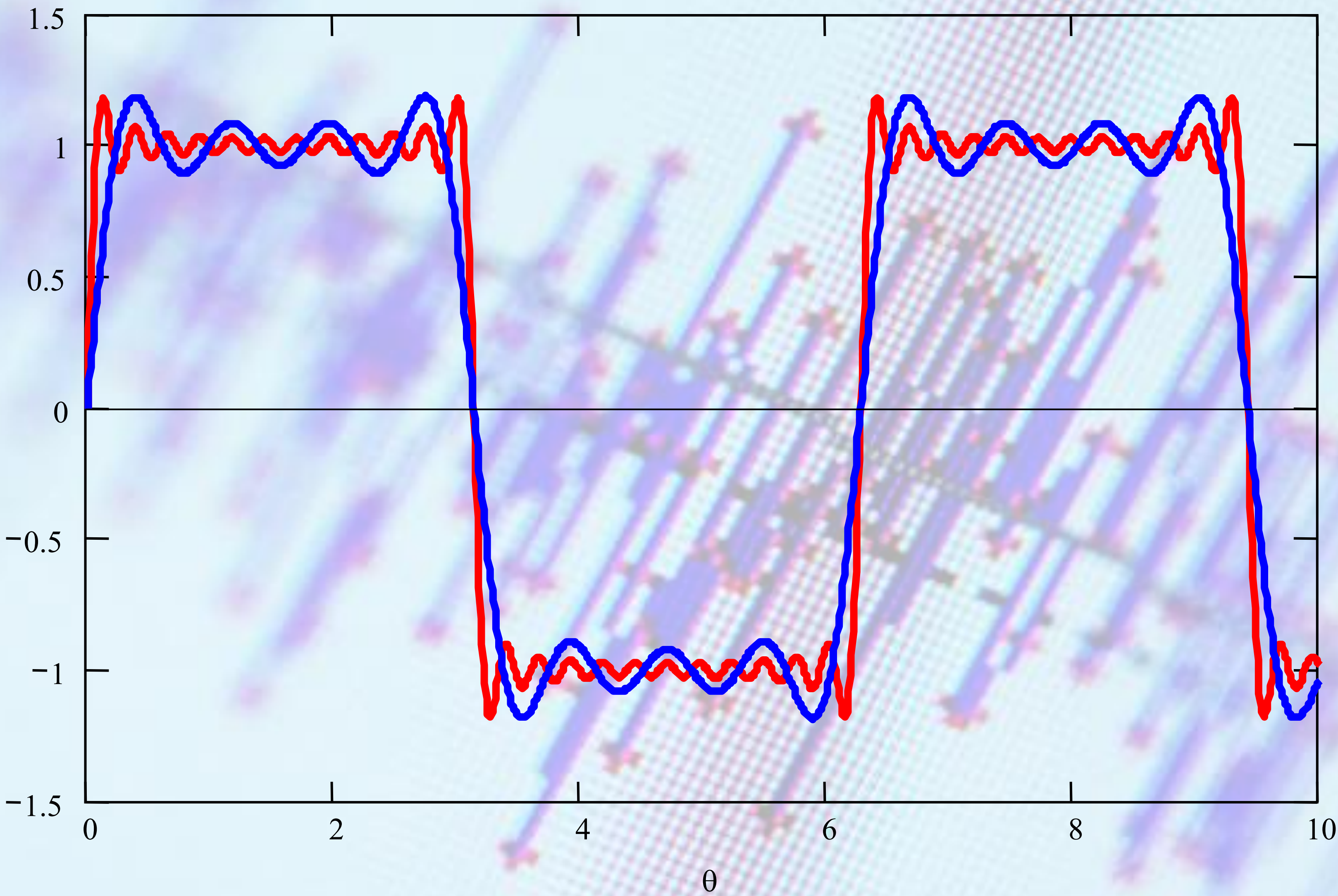
When we consider 12 terms, the function looks like the following.



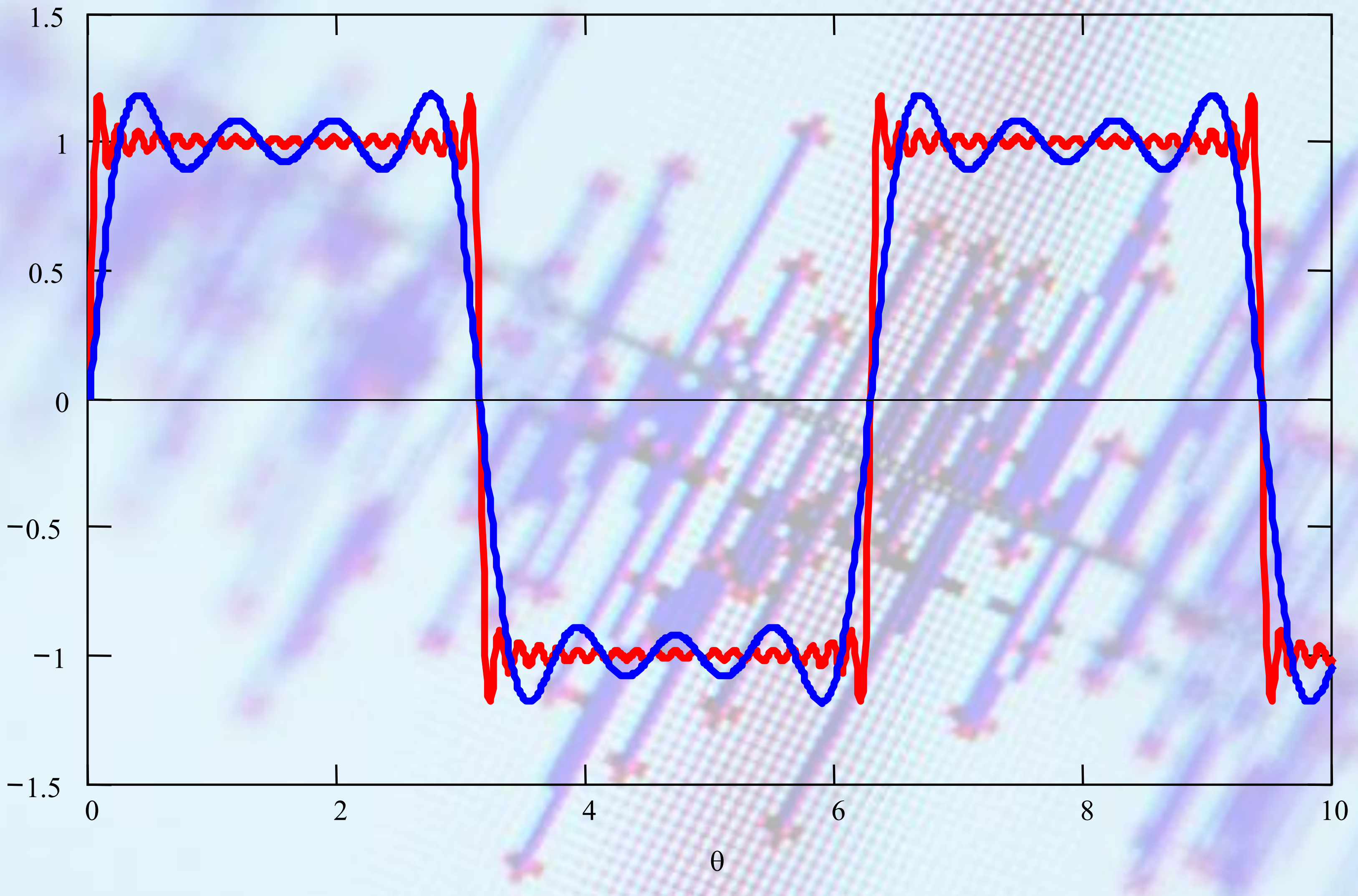
The red curve was drawn with 12 terms and the blue curve was drawn with 4 terms.



The red curve was drawn with 12 terms and the blue curve was drawn with 4 terms.



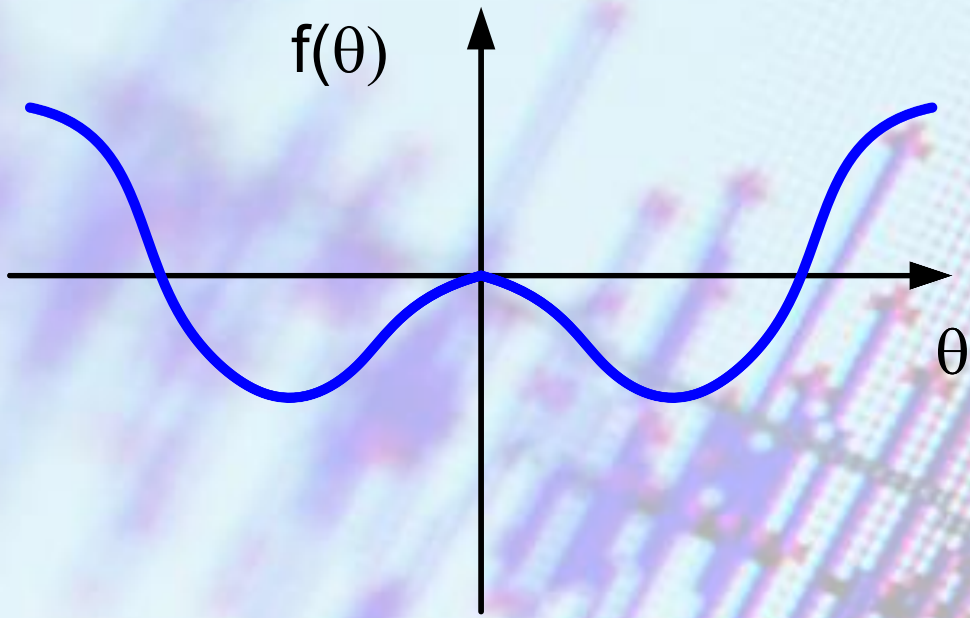
The red curve was drawn with 20 terms and the blue curve was drawn with 4 terms.



Even and Odd Functions

(We are not talking about even or odd numbers.)

Even Functions

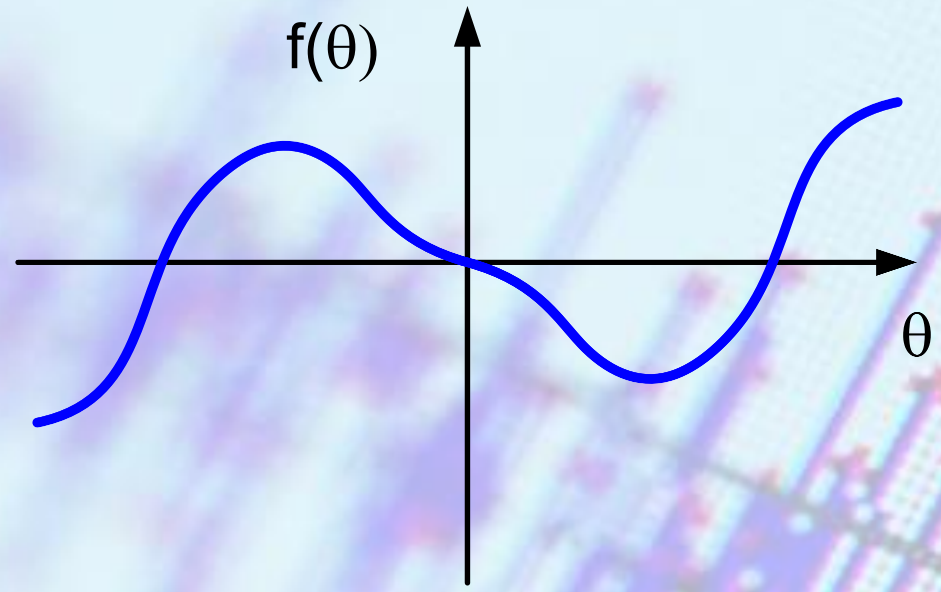


The value of the function would be the same when we walk equal distances along the X-axis in opposite directions.

Mathematically speaking -

$$f(-\theta) = f(\theta)$$

Odd Functions

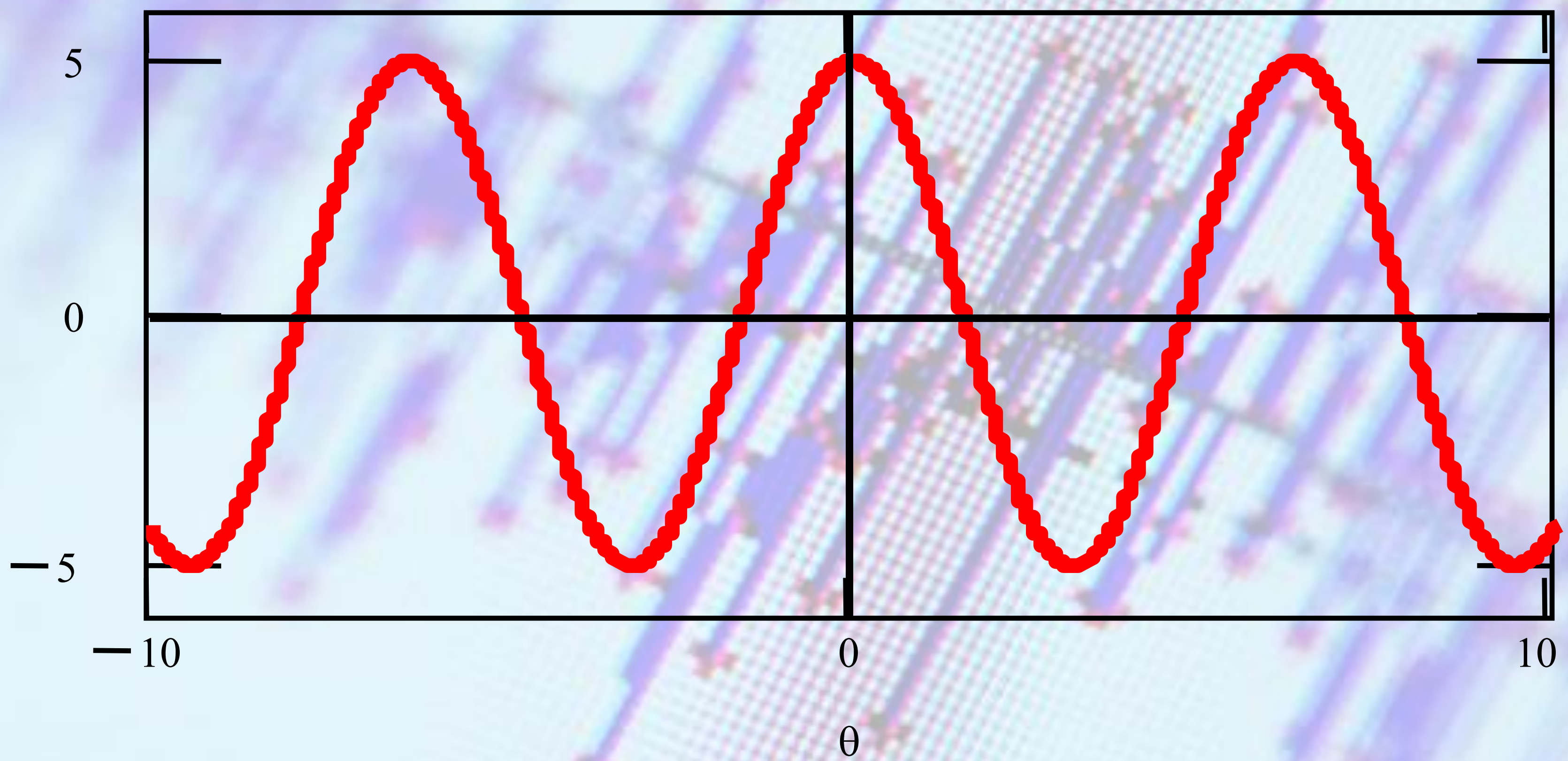


The value of the function would change its sign but with the same magnitude when we walk equal distances along the X-axis in opposite directions.

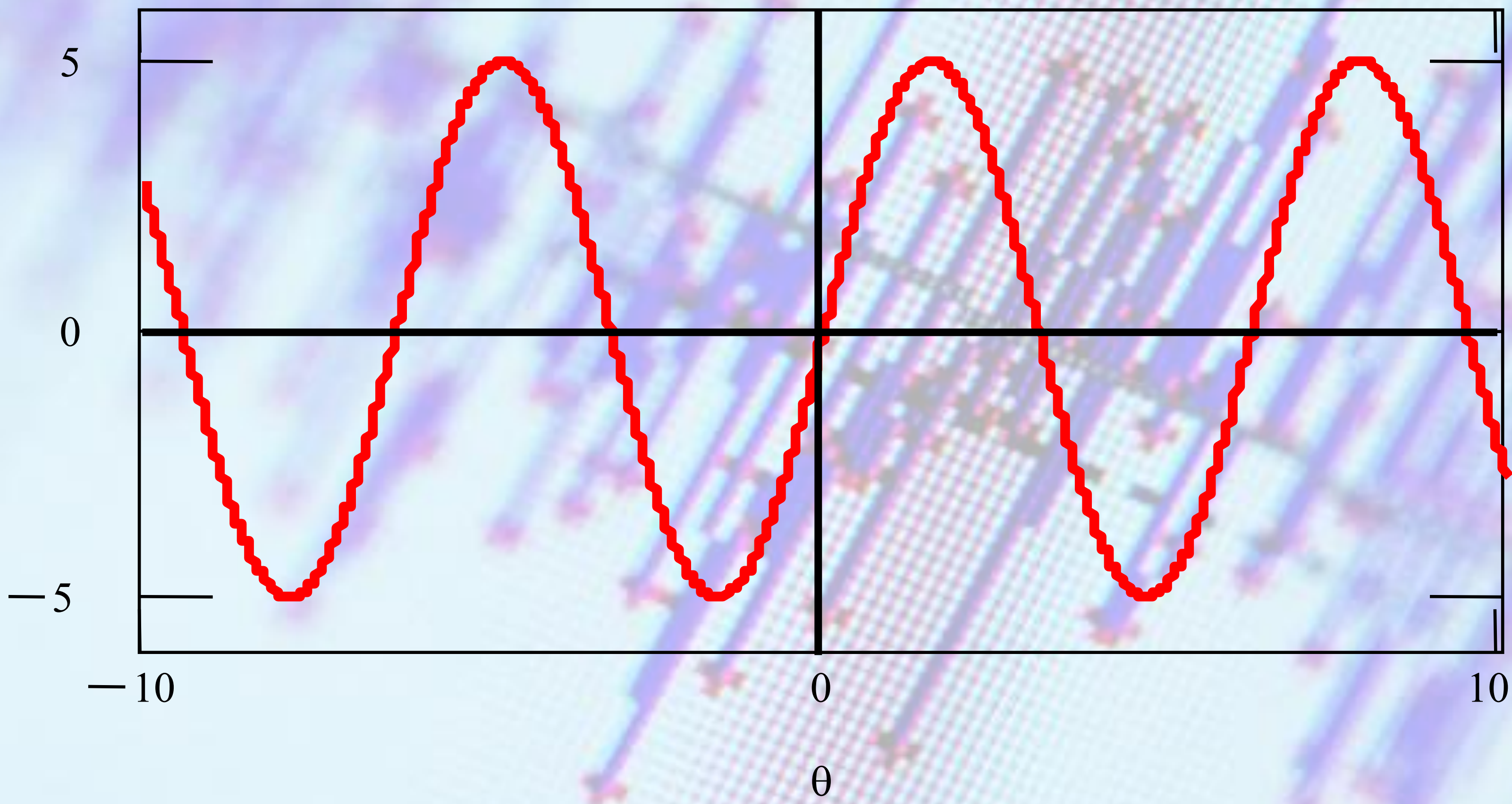
Mathematically speaking -

$$f(-\theta) = -f(\theta)$$

Even functions can solely be represented by cosine waves because, cosine waves are even functions. A sum of even functions is another even function.



Odd functions can solely be represented by sine waves because, sine waves are odd functions. A sum of odd functions is another odd function.



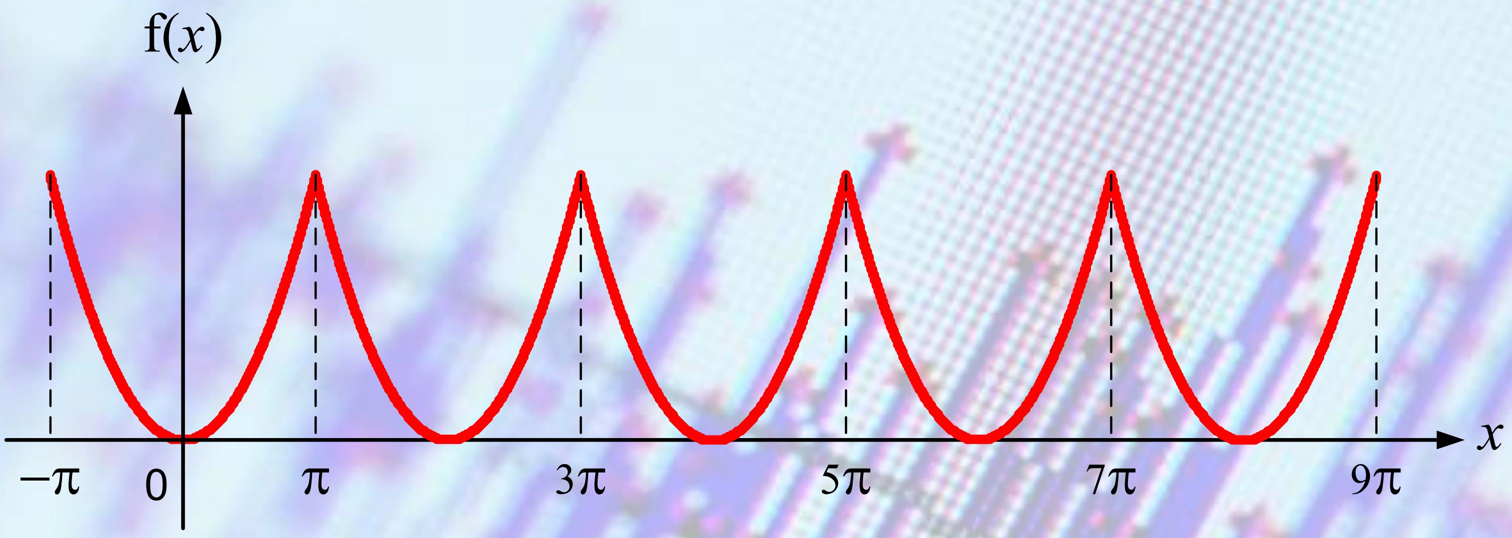
The Fourier series of an even function $f(\theta)$ is expressed in terms of a cosine series.

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

The Fourier series of an odd function $f(\theta)$ is expressed in terms of a sine series.

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

Example 2. Find the Fourier series of the following periodic function.



$$f(x) = x^2 \quad when \quad -\pi \le x \le \pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{x=-\pi}^{x=\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x^2 \cos nx dx\right]$$

Use integration by parts. Details are shown in your class note.

$$a_n = \frac{4}{n^2}\cos n\pi$$

$$a_n = -\frac{4}{n^2}$$ **when n is odd**

$$a_n = \frac{4}{n^2}$$ **when n is even**

This is an even function.

Therefore, $b_n = 0$

The corresponding Fourier series is

$$\frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right)$$

Week 12

Slide 234-259

Functions Having Arbitrary Period

Assume that a function $f(t)$ has period, T . We can relate angle (θ) with time (t) in the following manner.

$$\theta = \omega t$$

ω is the angular velocity in radians per second.

$$\omega = 2\pi f$$

f is the frequency of the periodic function, $f(t)$

$$\theta = 2\pi f t$$ where $$f = \frac{1}{T}$$

Therefore, $$\theta = \frac{2\pi}{T} t$$

$$\theta = \frac{2\pi}{T} t \qquad d\theta = \frac{2\pi}{T} dt$$

Now change the limits of integration.

$$\theta = -\pi \qquad -\pi = \frac{2\pi}{T} t \qquad t = -\frac{T}{2}$$

$$\theta = \pi \qquad \pi = \frac{2\pi}{T} t \qquad t = \frac{T}{2}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta \quad n = 1, 2, \cdots$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2\pi n}{T}t\right)dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi n}{T}t\right)dt \qquad n = 1, 2, \cdots$$

Example 3. Find the Fourier series of the following periodic function.

$$f(t) = t \quad when - \frac{T}{4} \le t \le \frac{T}{4}$$

$$= -t + \frac{T}{2} \quad when \quad \frac{T}{4} \le t \le \frac{3T}{4}$$

$$f(t+T)=f(t)$$

This is an odd function. Therefore, $a_n = 0$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{4}} t\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$+\frac{4}{T}\int_{\frac{T}{4}}^{\frac{T}{2}}\left(-t+\frac{T}{2}\right)\sin\left(\frac{2\pi n}{T}t\right)dt$$

Use integration by parts.

$$b_n = \frac{4}{T}\left[2.\left(\frac{T}{2\pi n}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{2T}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = 0$$ when *n* is even.

Therefore, the Fourier series is

$$\frac{2T}{\pi^2}\left[\sin\left(\frac{2\pi}{T}t\right)-\frac{1}{3^2}\sin\left(\frac{6\pi}{T}t\right)+\frac{1}{5^2}\sin\left(\frac{10\pi}{T}t\right)-\cdots\right]$$

The Complex Form of Fourier Series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Let us utilize the Euler formulae.

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2i}$$

The nth harmonic component of (1) can be expressed as:

$$a_n \cos n\theta + b_n \sin n\theta$$

$$= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} + b_n \frac{e^{jn\theta} - e^{-jn\theta}}{2i}$$

$$= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} - ib_n \frac{e^{jn\theta} - e^{-jn\theta}}{2}$$

$$a_n \cos n\theta + b_n \sin n\theta$$

$$= \left(\frac{a_n - jb_n}{2}\right)e^{jn\theta} + \left(\frac{a_n + jb_n}{2}\right)e^{-jn\theta}$$

Denoting

$$c_n = \left(\frac{a_n - jb_n}{2}\right), \quad c_{-n} = \left(\frac{a_n + jb_n}{2}\right)$$

and $c_0 = a_0$

$$a_n \cos n\theta + b_n \sin n\theta$$

$$= c_n e^{jn\theta} + c_{-n} e^{-jn\theta}$$

The Fourier series for $f(\theta)$ can be expressed as:

$$f(\theta) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jn\theta} + c_{-n} e^{-jn\theta}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

The coefficients can be evaluated in the following manner.

$$c_n = \left(\frac{a_n - jb_n}{2} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta - \frac{j}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)(\cos n\theta - j \sin n\theta) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-jn\theta} d\theta$$

$$c_{-n} = \left(\frac{a_n + jb_n}{2} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta + \frac{j}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)(\cos n\theta + j \sin n\theta) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{jn\theta} d\theta$$

$$c_n = \left(\frac{a_n - jb_n}{2} \right) \quad c_{-n} = \left(\frac{a_n + jb_n}{2} \right)$$

Note that c_{-n} is the complex conjugate of c_n . Hence we may write that

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-jn\theta} d\theta$$

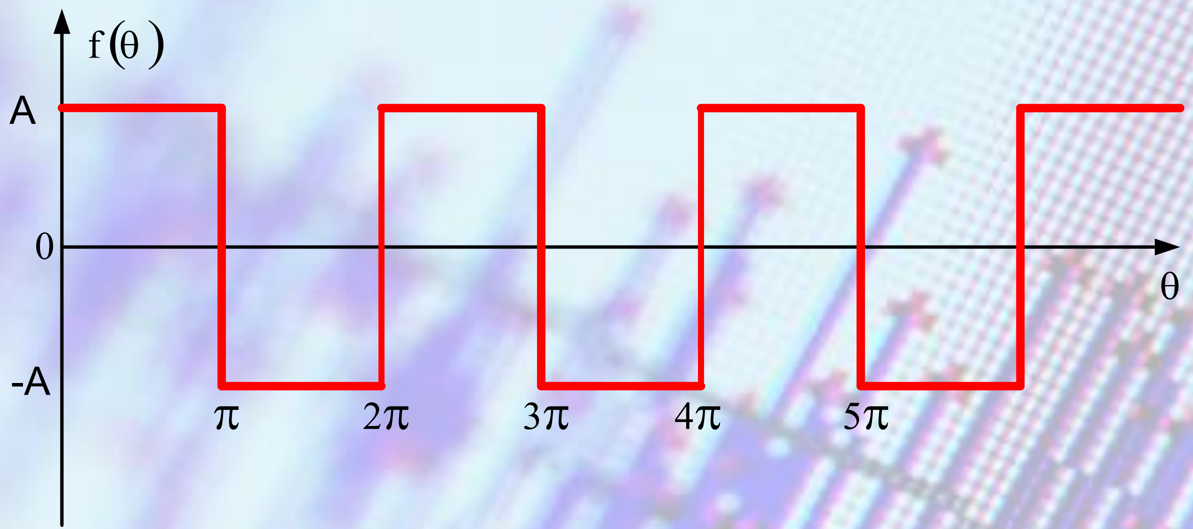
.

$$n = 0, \pm 1, \pm 2, \cdots$$

The complex form of the Fourier series of $f(\theta)$ with period 2π is:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

Example 1. Find the Fourier series of the following periodic function.



$$f(\theta) = A \quad when \quad 0 < \theta < \pi$$

$$= -A \quad when \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$A := 5$$

$$f(x) := \begin{cases} A & \text{if } 0 \le x < \pi \\ -A & \text{if } \pi \le x \le 2\cdot\pi \\ 0 & \text{otherwise} \end{cases}$$

$$A_0 := \frac{1}{2\pi} \cdot \int_0^{2\pi} f(x)\,dx$$

$$A_0 = 0$$

$$n := 1..8$$

$$A_n := \frac{1}{\pi} \cdot \int_0^{2\pi} f(x) \cdot \cos(n \cdot x)\, dx$$

$A_1 = 0$	$A_2 = 0$	$A_3 = 0$	$A_4 = 0$
$A_5 = 0$	$A_6 = 0$	$A_7 = 0$	$A_8 = 0$

$$B_n := \frac{1}{\pi} \cdot \int_0^{2\pi} f(x) \cdot \sin(n \cdot x)\, dx$$

$B_1 = 6.366$	$B_2 = 0$	$B_3 = 2.122$	$B_4 = 0$
$B_5 = 1.273$	$B_6 = 0$	$B_7 = 0.909$	$B_8 = 0$

Complex Form

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-jn\theta} d\theta$$

$$n = 0, \pm 1, \pm 2, \cdots$$

$$C(n) := \frac{1}{2\,\pi} \cdot \int_{0}^{2\,\pi} f(x) \cdot e^{-1i \cdot n \cdot x}\, dx$$

$$C(n) := \frac{1}{2\pi} \cdot \int_{0}^{2\pi} f(x) \cdot e^{-1i \cdot n \cdot x}\, dx$$

$C(0) = 0$	$C(1) = -3.183i$	$C(2) = 0$	$C(3) = -1.061i$
$C(4) = 0$	$C(5) = -0.637i$	$C(6) = 0$	$C(7) = -0.455i$

	$C(-1) = 3.183i$	$C(-2) = 0$	$C(-3) = 1.061i$
$C(-4) = 0$	$C(-5) = 0.637i$	$C(-6) = 0$	$C(-7) = 0.455i$

Week 13

Slide 261-278

Laplace Transform

Outline

- In this talk, we will:
  - Definition of the Laplace transform
  - A few simple transforms
  - Rules
  - Demonstrations

Laplace Transform
Background

- Classical differential equations

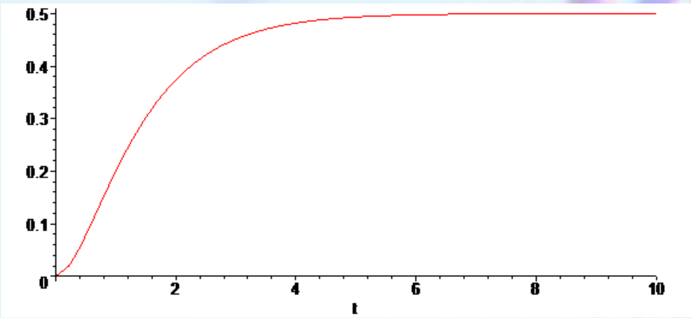
Time Domain

$$y^{(2)}(t)+y^{(1)}(t)+y(t)=x(t)$$

$$x(t)=1$$

Solve differential equation

$$y(t)=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}$$



Laplace Transform

Background

- Laplace transforms

Time Domain

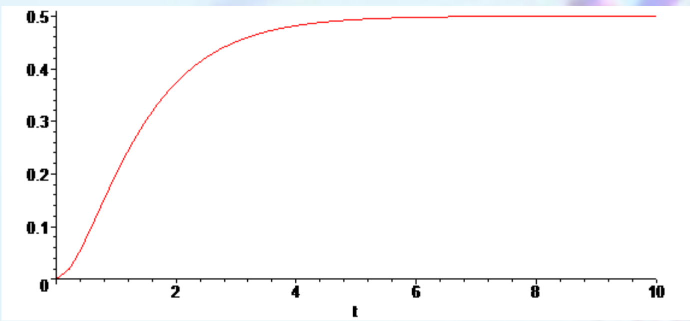
Frequency Domain

$$y^{(2)}(t)+y^{(1)}(t)+y(t)=x(t) \quad \xrightarrow{\text{Laplace transform}} \quad H(s)=\frac{1}{s^2+3s+2}$$

$$x(t)=1 \qquad X(t)=\frac{1}{s}$$

Solve algebraic equation

$$y(t)=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t} \quad \xleftarrow{\text{Inverse Laplace transform}} \quad \frac{1}{s}\frac{1}{s^2+3s+2}$$



Laplace Transform

Definition

- The Laplace transform is

$$\mathcal{L}\{\mathrm{f}(t)\} = \int_{0^-}^{\infty} \mathrm{f}(t) e^{-st} \, dt$$

$$= \mathrm{F}(s)$$

- Common notation:

$$\mathcal{L}\{\mathrm{f}(t)\} = \mathrm{F}(s) \qquad \mathrm{f}(t) \Leftrightarrow \mathrm{F}(s)$$

$$\mathcal{L}\{\mathrm{g}(t)\} = \mathrm{G}(s) \qquad \mathrm{g}(t) \Leftrightarrow \mathrm{G}(s)$$

Laplace Transform
Definition

- Notation:
  - Variables in italics t, s
  - Functions in time space f, g
  - Functions in frequency space F, G
  - Specific limits

$$\mathrm{f}\left(0^{+}\right)=\lim_{t\to 0^{+}} \mathrm{f}(t)$$

$$\mathrm{f}\left(0^{-}\right)=\lim_{t\to 0^{-}} \mathrm{f}(t)$$

Laplace Transform
Existence

- The Laplace transform of $\mathrm{f}(t)$ exists if
  - The function $\mathrm{f}(t)$ is piecewise continuous
  - The function is bound by $\left|\mathrm{f}(t)\right| \leq Me^{-kt}$ for some k and M

Laplace Transform
Example Transforms

- We will look at the Laplace transforms of:
  - The impulse function $\delta(t)$
  - The unit step function $\mathrm{u}(t)$
  - The ramp function t and monomials t^n
  - Polynomials, Taylor series, and e^t
  - Sine and cosine

Laplace Transform

Example Transforms

- While deriving these, we will examine certain properties:
  - Linearity
  - Damping
  - Time scaling
  - Time differentiation
  - Frequency differentiation

Laplace Transform

Impulse Function

- The easiest transform is that of the impulse function:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t) e^{-st} dt$$

$$= e^{-s \cdot 0}$$

$$= 1$$

$$\therefore \delta(t) \Leftrightarrow 1$$

Laplace Transform

Unit Step Function

- Next is the unit step function

$$\mathcal{L}\{\mathrm{u}(t)\} = \int_{0^-}^{\infty} \mathrm{u}(t) e^{-st}\, dt$$

$$\mathrm{u}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 1 \end{cases}$$

$$= \int_{0^-}^{\infty} e^{-st}\, dt$$

$$\therefore \mathrm{u}(t) \Leftrightarrow \frac{1}{s}$$

$$= -\frac{1}{s} e^{-st} \Big|_{0^-}^{\infty}$$

$$= -0 - \left(-\frac{1}{s} e^{-s \cdot 0} \right)$$

$$= \frac{1}{s}$$

Laplace Transform

Integration by Parts

- Further cases require integration by parts
- Usually written as

$$\int_a^b \mathrm{f}\,\mathrm{dg} = \mathrm{fg}\Big|_a^b - \int_a^b \mathrm{g}\,\mathrm{df}$$

Laplace Transform

Integration by Parts

- Product rule

$$\frac{d}{dt}\left(\mathrm{f}(t)\mathrm{g}(t)\right)=\left(\frac{d}{dt}\mathrm{f}(t)\right)\mathrm{g}(t)+\mathrm{f}(t)\left(\frac{d}{dt}\mathrm{g}(t)\right)$$

- Rearrange and integrate

$$\mathrm{f}(t)\left(\frac{d}{dt}\mathrm{g}(t)\right)=\frac{d}{dt}\left(\mathrm{f}(t)\mathrm{g}(t)\right)-\left(\frac{d}{dt}\mathrm{f}(t)\right)\mathrm{g}(t)$$

$$\int_a^b \mathrm{f}(t)\left(\frac{d}{dt}\mathrm{g}(t)\right)dt=\int_a^b \frac{d}{dt}\left(\mathrm{f}(t)\mathrm{g}(t)\right)dt-\int_a^b\left(\frac{d}{dt}\mathrm{f}(t)\right)\mathrm{g}(t)dt$$

$$=\mathrm{f}(t)\mathrm{g}(t)\Big|_a^b-\int_a^b\left(\frac{d}{dt}\mathrm{f}(t)\right)\mathrm{g}(t)dt$$

Ramp Function

- The ramp function

$$\mathcal{L}\{t\,\mathrm{u}(t)\} = \int_{0^-}^{\infty} t e^{-st} dt$$

$$\mathrm{f} = t$$

$$\mathrm{df} = \mathrm{d}t$$

$$\mathrm{dg} = e^{-st}\,\mathrm{d}t$$

$$\mathrm{g} = -\frac{1}{s} e^{-st}$$

$$= t\left(-\frac{1}{s} e^{-st}\right)\Bigg|_{0^-}^{\infty} - \int_{0^-}^{\infty} 1\left(-\frac{1}{s} e^{-st}\right) dt$$

$$= 0 + \frac{1}{s}\int_{0^-}^{\infty} e^{-st} dt$$

$$= \frac{1}{s}\left(-\frac{1}{s} e^{-st}\right)\Bigg|_{0^-}^{\infty} = \frac{1}{s^2}$$

$$\therefore t\,\mathrm{u}(t) \Leftrightarrow \frac{1}{s^2}$$

Laplace Transform

Monomials

- By repeated integration-by-parts, it is possible to find the formula for a general monomial for $n \geq 0$

$$\mathcal{L}\left\{t^n \, \mathrm{u}(t)\right\} = \frac{n!}{s^{n+1}}$$

$$\therefore t^n \, \mathrm{u}(t) \Leftrightarrow \frac{n!}{s^{n+1}}$$

Laplace Transform

Linearity Property

- The Laplace transform is linear
- If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then

$$\mathcal{L}\{a\,f(t) + b\,g(t)\} = a\,F(s) + b\,G(s)$$

$$a\,f(t) + b\,g(t) \Leftrightarrow a\,F(s) + b\,G(s)$$

Laplace Transform

Initial and Final Values

- Given $f(t) \Leftrightarrow F(s)$ then

$$f(0^+) = \lim_{s\to\infty} s\,F(s)$$

$$f(\infty) = \lim_{s\to 0^+} s\,F(s)$$

- Note $sF(s)$ is the Laplace transform of $f^{(1)}(x)$

Laplace Transform
Polynomials

- The Laplace transform of the polynomial follows:

$$\mathcal{L}\left\{\sum_{k=0}^{n} a_k t^k \, \mathrm{u}(t)\right\} = \sum_{k=0}^{n} a_k \frac{k!}{s^{k+1}}$$

Laplace Transform
Polynomials

- This generalizes to Taylor series, *e.g.*,

$$\mathcal{L}\left\{e^{t}\,\mathrm{u}(t)\right\}=\mathcal{L}\left\{\sum_{k=0}^{n}\frac{1}{k!}t^{k}\,\mathrm{u}(t)\right\}$$

$$=\sum_{k=0}^{n}\frac{1}{k!}\frac{k!}{s^{k+1}}$$

$$=\sum_{k=0}^{n}\frac{1}{s^{k+1}}$$

$$=\frac{1}{s-1}$$

$$\therefore e^{t}\,\mathrm{u}(t)\Leftrightarrow\frac{1}{s-1}$$

Week 14

Slide 280-289

The Sine Function

- Sine requires two integration by parts:

$$\mathcal{L}\{\sin(t)\mathrm{u}(t)\} = \int_{0^-}^{\infty} \sin(t)e^{-st}\,dt$$

$$= -\frac{1}{s}\sin(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} -\frac{1}{s}\cos(t)e^{-st}\,dt$$

$$= 0 + \frac{1}{s}\int_{0^-}^{\infty} \cos(t)e^{-st}\,dt$$

$$= -\frac{1}{s^2}\cos(t)e^{-st}\Big|_{0^-}^{\infty} - \frac{1}{s}\int_{0^-}^{\infty} \frac{1}{s}\sin(t)e^{-st}\,dt$$

$$= \frac{1}{s^2} - \frac{1}{s^2}\int_{0^-}^{\infty} \sin(t)e^{-st}\,dt$$

$$= \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}\{\sin(t)\mathrm{u}(t)\}$$

1 of 2

The Sine Function

- Consequently:

$$\mathcal{L}\{\sin(t)\mathrm{u}(t)\} = \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}\{\sin(t)\mathrm{u}(t)\}$$

$$(s^2+1)\mathcal{L}\{\sin(t)\mathrm{u}(t)\} = 1$$

$$\mathcal{L}\{\sin(t)\mathrm{u}(t)\} = \frac{1}{s^2+1}$$

$$\therefore \sin(t)\mathrm{u}(t) \Leftrightarrow \frac{1}{s^2+1}$$

2 of 2

Laplace Transform
The Cosine Function

- As does cosine:

$$\mathcal{L}\{\cos(t)\mathrm{u}(t)\} = \int_{0^-}^{\infty} \cos(t) e^{-st}\, dt$$

$$= -\frac{1}{s}\cos(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{1}{s}\sin(t)e^{-st}\, dt$$

$$= \frac{1}{s} - \frac{1}{s}\int_{0^-}^{\infty} \sin(t)e^{-st}\, dt$$

$$= \frac{1}{s} - \left(-\frac{1}{s^2}\sin(t)e^{-st}\Big|_{0^-}^{\infty} - \frac{1}{s}\int_{0^-}^{\infty} -\frac{1}{s}\cos(t)e^{-st}\, dt \right)$$

$$= \frac{1}{s^2} + 0 - \frac{1}{s^2}\int_{0^-}^{\infty} \cos(t)e^{-st}\, dt$$

$$= \frac{1}{s} - \frac{1}{s^2}\mathcal{L}\{\cos(t)\mathrm{u}(t)\}$$

1 of 2

The Cosine Function

- Consequently:

$$\mathcal{L}\{\cos(t)\mathrm{u}(t)\} = \frac{1}{s} - \frac{1}{s^2}\mathcal{L}\{\cos(t)\mathrm{u}(t)\}$$

$$\left(s^2+1\right)\mathcal{L}\{\cos(t)\mathrm{u}(t)\} = s$$

$$\mathcal{L}\{\cos(t)\mathrm{u}(t)\} = \frac{s}{s^2+1}$$

$$\therefore \cos(t)\mathrm{u}(t) \Leftrightarrow \frac{s}{s^2+1}$$

2 of 2

Laplace Transform

Damping Property

- Time domain damping ⇔ frequency domain shifting

$$\mathcal{L}\left\{e^{-at}\,\mathrm{f}(t)\right\} = \int_{0^-}^{\infty} e^{-at}\,\mathrm{f}(t)e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \mathrm{f}(t)e^{-(s+a)t}\,dt$$

$$= \mathrm{F}(s+a)$$

$$\therefore e^{-at}\,\mathrm{f}(t) \Leftrightarrow \mathrm{F}(s+a)$$

Laplace Transform

Damping Property

- Damped monomials

$$t^n \, \mathrm{u}(t) \Leftrightarrow \frac{n!}{s^{n+1}}$$

$$e^{-at} t^n \, \mathrm{u}(t) \Leftrightarrow \frac{n!}{(s+a)^{n+1}}$$

A special case:

$$\mathrm{u}(t) \Leftrightarrow \frac{1}{s}$$

$$e^{-at} \, \mathrm{u}(t) \Leftrightarrow \frac{1}{s+a}$$

Laplace Transform

Time-Scaling Property

- Time domain scaling ⇔ attenuated frequency domain scaling

$$\mathcal{L}\{\mathrm{f}(at)\} = \int_{0^-}^{\infty} \mathrm{f}(at)e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \mathrm{f}(\tau)e^{-s\frac{\tau}{a}}\frac{1}{a}\,d\tau$$

$$= \frac{1}{a}\int_{0^-}^{\infty} \mathrm{f}(\tau)e^{-\left(\frac{s}{a}\right)\tau}\,ad\tau$$

$$= \frac{1}{a}\mathrm{F}\left(\frac{s}{a}\right)$$

$$\tau = at$$

$$\frac{d\tau}{a} = dt$$

$$\therefore \mathrm{f}(at) \Leftrightarrow \frac{1}{a}\mathrm{F}\left(\frac{s}{a}\right)$$

Laplace Transform
Time-Scaling Property

- Time scaling of trigonometric functions:

$$\sin(t)\mathrm{u}(t) \Leftrightarrow \frac{1}{s^2+1} \qquad \cos(t)\mathrm{u}(t) \Leftrightarrow \frac{s}{s^2+1}$$

$$\mathcal{L}\{\sin(\omega t)\mathrm{u}(t)\} = \frac{1}{\omega}\frac{1}{\left(\frac{s}{\omega}\right)^2+1} \qquad \mathcal{L}\{\cos(\omega t)\mathrm{u}(t)\} = \frac{1}{\omega}\frac{\left(\frac{s}{\omega}\right)}{\left(\frac{s}{\omega}\right)^2+1}$$

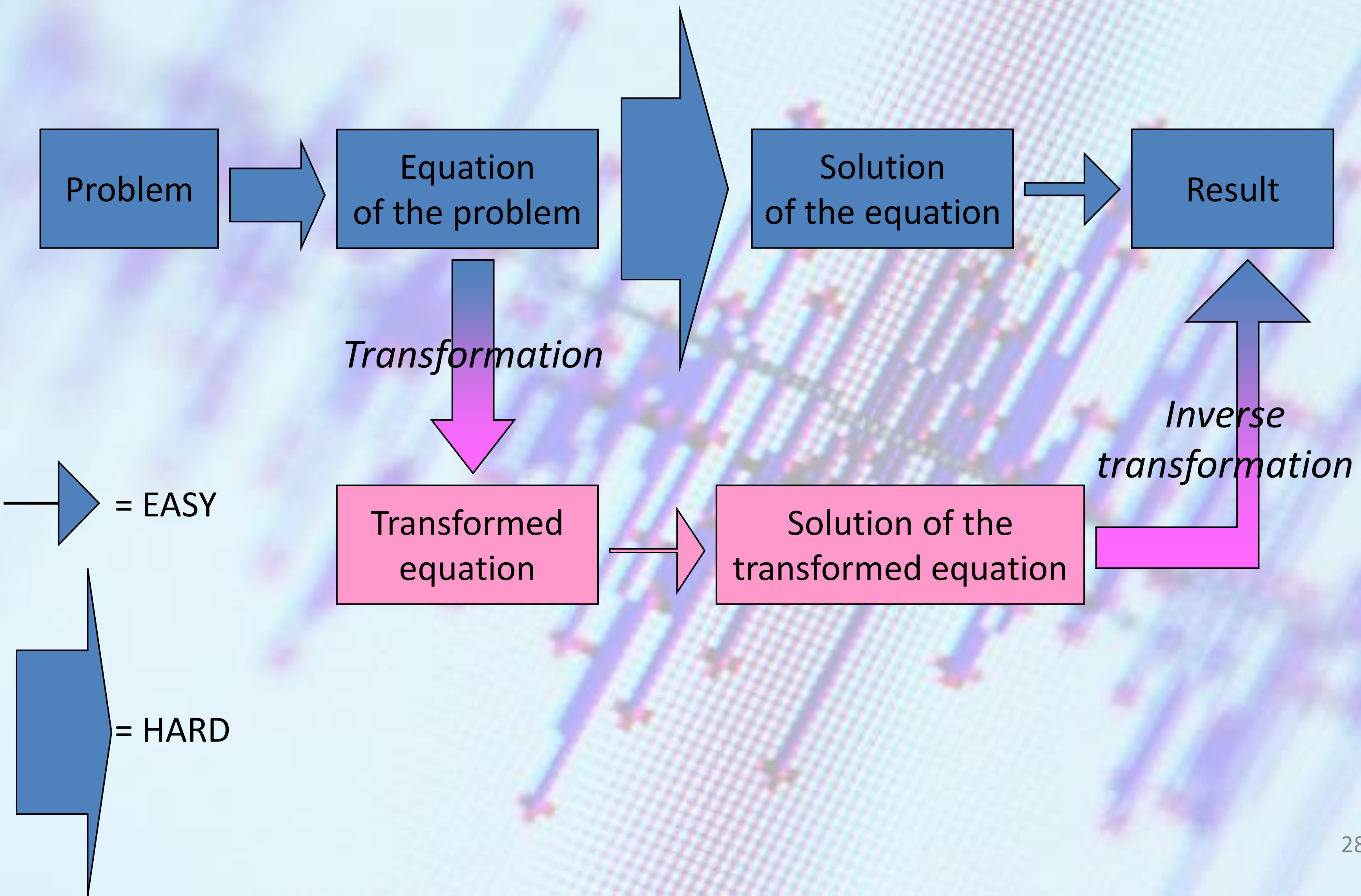
$$= \frac{\omega}{s^2+\omega^2} \qquad = \frac{s}{s^2+\omega^2}$$

$$\therefore \sin(\omega t)\mathrm{u}(t) \Leftrightarrow \frac{\omega}{s^2+\omega^2} \qquad \cos(\omega t)\mathrm{u}(t) \Leftrightarrow \frac{s}{s^2+\omega^2}$$

Why use Transforms?

- Transforms are not simply math curiosity sketched at the corner of a woodstove by ol' Frenchmen.
- Way to reframe a problem in a way that makes it easier to understand, analyze and solve.

General Scheme using Transforms

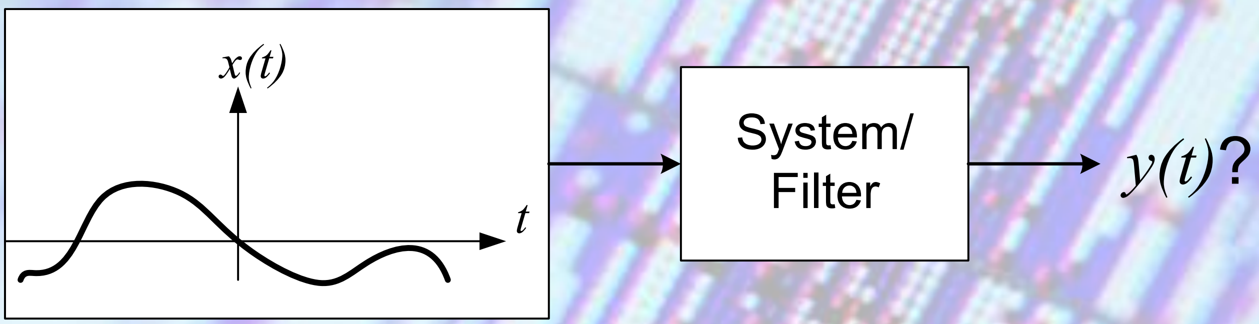


Week 15

Slide 291-309

Typical Problem

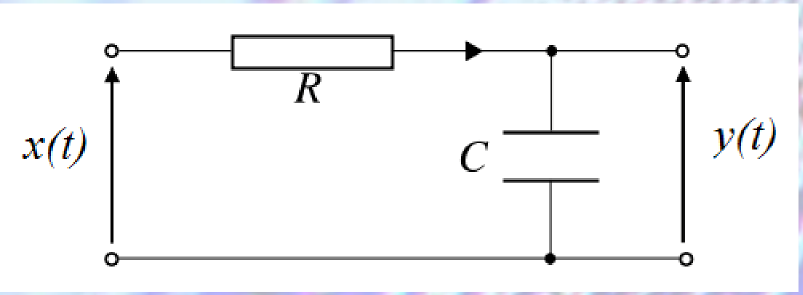
- Given an input signal $x(t)$, what is the output signal $y(t)$ after going through the system?



- To solve it in the <u>*time domain*</u> (t) is cumbersome!

Integrating Differential Equation?

- Let's have a simple first order low-pass filter with resistor R and capacitor C:



- The system is described by diff. eq.:

$$RCy'(t) + y(t) = x(t)$$

- To find a solution, we can integrate. *Ugh!*

Laplace Transform

- Formal definition:

$$\mathbf{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

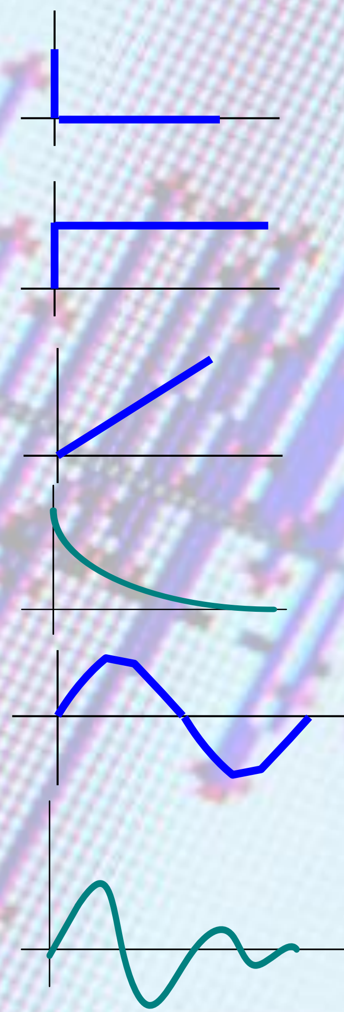
- Compare this to FT:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

- Small differences:
  - *Integral from 0 to ∞ to for Laplace*
    - *f(t) for t<0 is not taken into account*
  - *-s instead of -iω*

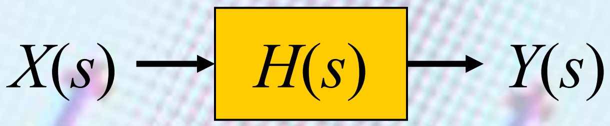
Common Laplace Transfom

Name	$f(t)$	$F(s)$
Impulse δ	$f(t) = \begin{cases} 1 & t = 0 \\ 0 & t > 0 \end{cases}$	1
Step	$f(t) = 1$	$\frac{1}{s}$
Ramp	$f(t) = t$	$\frac{1}{s^2}$
Exponential	$f(t) = e^{-at}$	$\frac{1}{s+a}$
Sine	$f(t) = \sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
Damped Sine	$f(t) = e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$



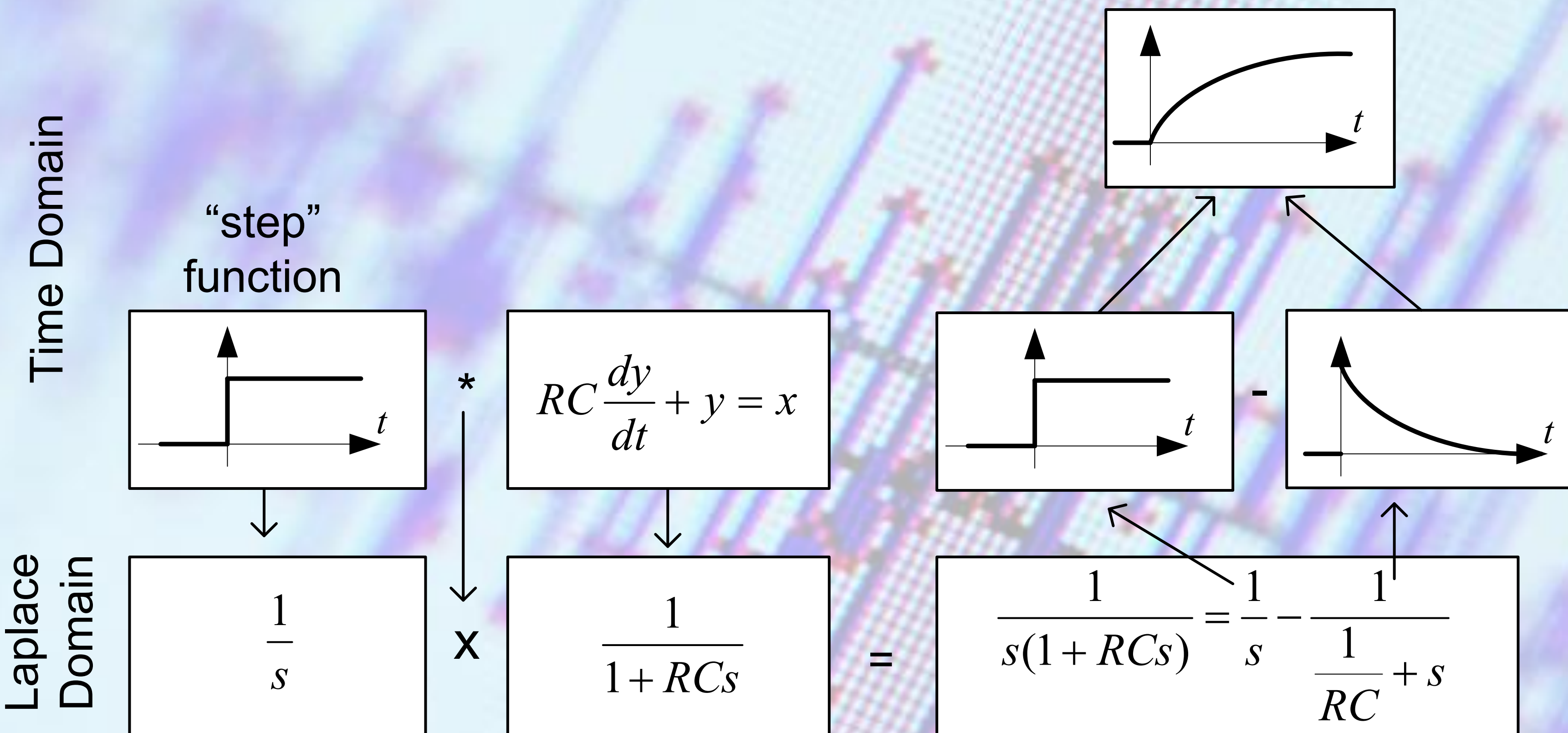
Transfer Function H(s)

- Definition
  - $H(s) = Y(s) / X(s)$

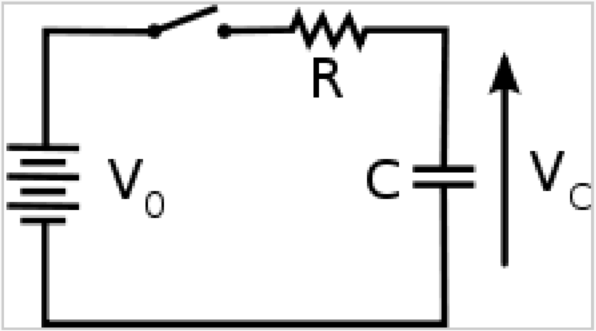


- Relates the output of a linear system (or component) to its input.
- Describes how a linear system responds to an impulse.
- All linear operations allowed
  - Scaling, addition, multiplication.

RC Circuit Revisited



$$I(t) = \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = C\frac{\mathrm{d}V(t)}{\mathrm{d}t}$$



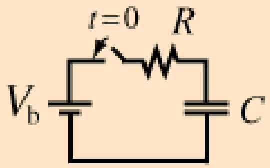
A simple resistor-capacitor circuit demonstrates charging of a capacitor.

Vo=V(t)+R I(t)

$$V_0 = v_{\text{resistor}}(t) + v_{\text{capacitor}}(t) = i(t)R + \frac{1}{C}\int_{t_0}^{t} i(\tau)\mathrm{d}\tau$$

Charging a Capacitor

When a battery is connected to a series resistor and capacitor, the initial current is high as the battery transports charge from one plate of the capacitor to the other. The charging current asymptotically approaches zero as the capacitor becomes charged up to the battery voltage. Charging the capacitor stores energy in the electric field between the capacitor plates. The rate of charging is typically described in terms of a time constant RC.



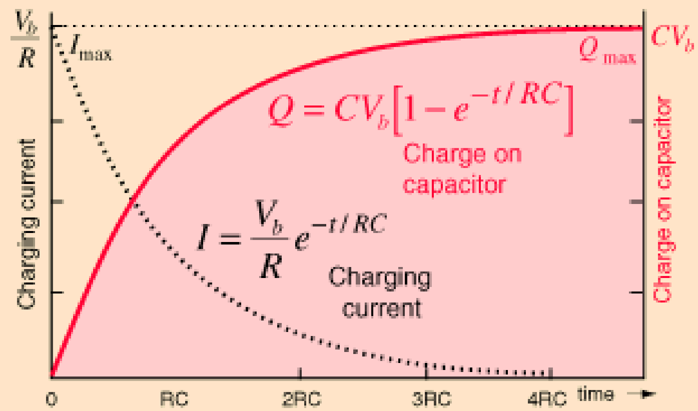
$$V_b = V_R + V_C$$

$$V_b = IR + \frac{Q}{C}$$

As charging progresses,

$$V_b = IR + \frac{Q}{C}$$

current decreases and charge increases.



At $t = 0$
$Q = 0$
$V_C = 0$
$I = \frac{V_b}{R}$

As $t \to \infty$
$Q \to CV_b$
$V_C \to V_b$
$I \to 0$

Calculation	Derive expressions	Capacitor discharge	Air tank analogy

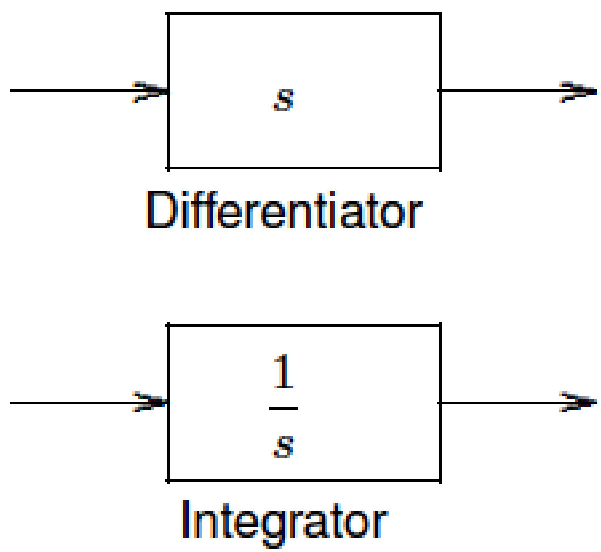
HyperPhysics***** Electricity and Magnetism *R Nave*

Laplace for Circuits

Very simply, the Laplace transform substitutes s, the Laplace transform operator for the differential operator d/dt. Then the s term may be manipulated like any other variable.

Thus one will see s in a control system block to indicate *differentiator* and $1/s$ to indicate *integrator*.

The substitution of s for d/dt leads to another one, s for $j\omega$. This is useful in determining the transfer function of an electrical network, and then its magnitude and phase responses.


s
Differentiator
$\frac{1}{s}$
Integrator

A Simple Example: Capacitor Charging Equation

$$i(t) = C\frac{d}{dt}v(t) \tag{1}$$

where:

$i(t)$	Current in the capacitor, amps, as a function of time
$v(t)$	Voltage across the capacitor, volts, as a function of time
C	Capacitance, farads

In words, the time-varying current into a capacitor is proportional to the rate of change of the voltage across its terminals. The constant of proportionality is the capacitance C.

To apply the Laplace transform to this equation, we replace the differential operator d/dt by s and the voltage and current by their transformed versions:

$$\begin{aligned} i(s) &= Csv(s) \\ &= sC\,v(s) \end{aligned} \tag{2}$$

where:

$i(s)$	Current in the capacitor, amps, in the Laplace domain
$v(s)$	Voltage across the capacitor, volts, in the Laplace domain
C	Capacitance, farads

Let's reverse this and solve for the capacitor voltage:

$$v(s) = \frac{1}{C}\frac{1}{s}i(s) \tag{3}$$

Now, back to the time domain: voltage and current transform to their time-dependent values and $1/s$ becomes an integral:

$$v(t) = \frac{1}{C}\int i(t)dt \tag{4}$$

This is the long way 'round. Equation 4 could be obtained directly by inspection of equation 1. However, it shows a very simple application of the Laplace transform: we transformed the original equation into the Laplace domain, manipulated it, and then transformed the result back into the time domain[5].

Inductor Differential Equation

A similar reasoning process can be applied to the inductor. The basic differential equation relating voltage and current in an inductor is:

$$v(t) = L\frac{di(t)}{dt} \tag{5}$$

where:

$v(t)$	Voltage across the inductor, volts
$i(t)$	Current in the inductor, amps
C	Inductance, Henries

Proceeding as we did in the case of capacitance, we replace the differential operator d/dt by s and the voltage and current by their transformed versions:

$$\begin{aligned} v(s) &= Lsi(s) \\ &= sL\, i(s) \end{aligned} \tag{6}$$

The reactance of the inductor (analogous to resistance, but affecting AC current only) is the ratio of voltage to current:

$$\frac{v(s)}{i(s)} = sL \qquad (7)$$

Inductors may then be represented by inductive reactance as sL_1, sL_2 and so on.

Transfer Function of Low Pass RC Filter

A simple RC lowpass filter is shown in figure 4, where the capacitor is indicated by its reactance $1/sC$. Let us determine the frequency response of this filter.

The frequency response shows the relationship between output voltage and input voltage as a function of frequency. Consequently, a first step is to determine the relationship between e_o and e_i. The resistor and the reactance of the capacitor form a voltage divider, so we can write:

$$\frac{e_o}{e_i} = \frac{Z_2}{Z_1 + Z_2} \qquad (10)$$

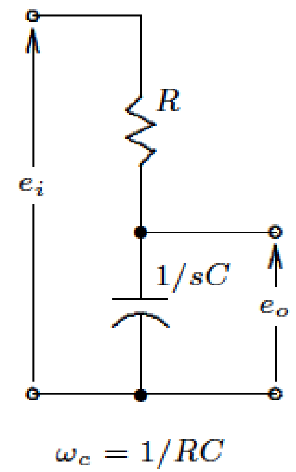


Figure 4: RC Lowpass Filter

where:

e_o	AC Output voltage voltage from the circuit
e_i	AC Input voltage to the circuit
Z_1	Impedance of the upper half of the voltage divider (the resistor)
Z_2	Impedance of the bottom half of the voltage divider (the capacitor)

Capacitive Reactance

The reactance of the capacitor is the ratio of voltage to current:

$$\frac{v(s)}{i(s)} = \frac{1}{sC} \tag{8}$$

It's common to represent the capacitive reactance directly on a circuit diagram, so one sees capacitors labelled as $1/sC_1$, $1/sC_2$ and so on. For the purpose of circuit analysis these reactances may be treated as resistances.

The magnitude phase of capacitive reactance are also represented as

$$X_c = \frac{1}{j\omega C} \tag{9}$$

where the variables are:

X_c	Capacitive reactance, ohms
j	Imaginary operator, $\sqrt{-1}$
ω	Circular frequency, radians/sec

Strictly speaking, e_o should be written as $e_o(\omega)$ or $e_o(f)$ to indicate that the value is a function of frequency, but we'll take that as understood.

Now substitute R for Z_1, $1/sC$ for Z_2 and do some algebra:

$$\begin{aligned} \frac{e_o}{e_i} &= \frac{1/sC}{R + 1/sC} \\ &= \frac{1}{1 + sRC} \end{aligned} \tag{11}$$

Now we need to introduce some new labels. The quantity RC is important in these circuits: it is known as the *time constant* τ and will turn up again when we look at the time-domain response of the filter.

$$\tau = RC \tag{12}$$

Then we could rewrite equation 11 this way:

$$\frac{e_o}{e_i} = \frac{1}{1 + s\tau} \tag{13}$$

We can do even better than this. In the frequency domain RC is related to the *corner* or *cutoff frequency* of the filter, which is referred to as ω_o in radians/sec notation or f_o in Hertz (cycles/second) .

$$\begin{aligned} \omega_o &= \frac{1}{\tau} \\ &= \frac{1}{RC} \end{aligned} \tag{14}$$

So equation 13 could be written as:

$$\frac{e_o}{e_i} = \frac{1}{1 + s/\omega_o} \tag{15}$$

Transfer Function of Low Pass LR Filter

The LR lowpass filter is shown in figure 7. Now we'll determine the frequency response of this filter. As we'll see, this is very similar to the RC lowpass filter of the previous section.

The inductor and resistor form a voltage divider, so we can write:

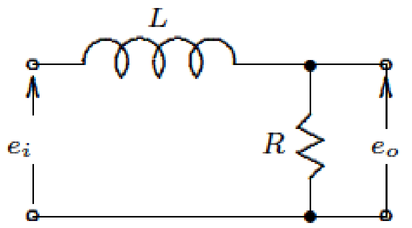


Figure 7: LR Lowpass Filter

$$\frac{e_o}{e_i} = \frac{Z_2}{Z_1 + Z_2} \tag{21}$$

where

e_o	AC Output voltage voltage from the circuit
e_i	AC Input voltage to the circuit
Z_1	Impedance of the upper half of the voltage divider (the inductor)
Z_2	Impedance of the bottom half of the voltage divider (the resistor)

Now substitute sL for Z_1, R for Z_2 and do some algebra:

$$\begin{aligned} \frac{e_o}{e_i} &= \frac{R}{R + sL} \\ &= \frac{1}{1 + s\frac{L}{R}} \end{aligned} \tag{22}$$